Spectrum of Mechanical Engineering and Operational Research

Journal homepage: www.smeor-journal.org
eISSN: 3042-0288

T-Bipolar Soft Semigroups and Related Results

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ARTICLE INFO

Article history:
Received 10 May 2024
Received in revised form 29 September 2024
Accepted 1 October 2024
Available online 3 October 2024

Keywords:
T-bipolar soft set; Semigroup; AND product; OR product; Extended union; Extended intersection.

ABSTRACT

The role of semigroups (SGs) is to operate as the tool for the study and modelling of systems and phenomena in which inverse operation is not relevant or necessary. SGs are encountered in the subject of automata theory, coding theory, language theory, and the study of discrete dynamical systems. Moreover, T-bipolar soft set is closer to bipolarity as compared to other theories on bipolar soft sets. Based on the idea of T-bipolar soft sets, here we explore the notion and properties of the T-bipolar soft semigroup (T-BSSG). These properties include the AND and OR product, the Restricted Union (Res-Union) and Restricted Intersection (Res-Intersection), and the Extended Union (Ext-Union) and Extended Intersection (Ext-Intersection) on T-BSSG. Further, we also devise the related algebraic properties of T-BSSG.

1. Introduction

The study of group and ring was a much earlier development, but the discovery of semigroups (SGs) was a late comer in this branch of algebra with a complex axiomatic set. Howie [1] explored the fundamental theory about the notion of SGs. The interpretation theory of SGs was devised by Shain [2] in 1963 by employing a binary relation on a set and for the SG product he employed the composition of relations. McKenzie and Schein [3] achieved the result that every SG is isomorphic to a transitive SG of binary relation. SGs are applied in many areas such as in different fields. In computer science, they are employed in the formal language study and automata theory where they serve as a mathematical framework for the analysis and comprehension of the behavior of finite-state machines. In the study of coding theory, semigroups are used in the development and investigation of the error-correcting codes. Moreover, semigroups are used in the study of algebraic structures like groups and monoids which are algebraic structures constructed on top of semigroups as they are more restrictive algebraic structures. It is much simpler to deal with semigroups than with other algebraic structures, and their study has resulted in numerous theoretical developments and practical applications in many areas of science and technology. Soft set (SS) theory was devised by a famous mathematician Molodtsov [4]. It has been interpreted as a modification of the crisp set

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https://doi.org/10.31181/smeor11202421

concept for the representation of uncertainties and imprecise data in a parametric manner. Unlike the FS theory that is a tool for assigning membership degrees to elements, the SS theory is a parameterized family of subsets in the universal set. This method is a strong and robust way to handle uncertainties. It can apply in various situations of the data analysis, decision making, and computer science. SS theory is not only universal and capable of dealing with various kinds of uncertainties and imprecisions, but it also does not require any extra quantitative measure as compared to probability theory and FS theory It is a process of presenting and handling data which is indefinite and uncertain, and thus, makes it applicable for different uses like decision making in inexact situations, data mining, and pattern recognition. SST is a theory that has been greatly improved since its initial formulation and has led to many generalizations and extensions. Ali *et al.,* [5] devised certain new operations on SS. The multi-attribute decision-making (MADM) strategy constructed around SST is developed by Zahedi Khameneh and Kılıçman [6]. Xiao *et al.,* [7] gave a brief introduction on the concepts of acknowledgment of soft data according to the SS theory. SST are widely used in many different sectors including in real life. The new extension and application of SST in the field of fuzzy mathematics are discussed by Tripathy *et al.,* [8] with the title of latest approach to fuzzy SST and its application in decision making (DM). Cagman *et al.,* [9] also discussed the fuzzy SST and its applications in different ways. Mushrif *et al.*, [10] presents an innovative method for classifying textures, SST based classification algorithm. Moreover, Min [11] developed the similarity in SST. The concept of financial ratio selection for SST-based company failure prediction is covered by Xu *et al.,* [12]. Moreover, Danjuma *et al.*, [13] gave a unique and helpful idea on a different strategy for the SST normal parameter reduction technique. Maji *et al.,* [14] provided the first definition of the term "soft subsets".

A soft semigroup (SSG) structure unites the ideas of SST and SG theory. It was proposed as an extension of SST to study algebraic structures with uncertainties or imperfections. SSGs give the possibility to describe algebraic structures including uncertainties or imprecisions in a parameterized way. They enable the depiction of semigroups-like structures, with their elements or operations being unclear or vague. The soft ordered SGs were first devised Jun *et al.,* [15]. They studied the basic characteristics of these systems, and subsequently this provided the framework for further advancement in this field. Feng *et al.,* [16] applied the soft relations to SGs. They were concerned with soft bonds and how they are important in SG theory. Khan *et al.,* [17] analyzed the concept of uni-soft structure for the ordered SGs. A new study was conducted and it provided an understanding of uni-soft structure and its usage in ordered SGs. Hamouda [18] studied soft ideals in ordered SGs. It enabled to unravel the role and importance of soft ideals in the realm of ordered SG theory. Shabir *et al.*, [19] studied soft ideals and generalized fuzzy ideals in SGs with a focus on exposing their properties and usages. Muhiuddin and Mahboob [20] devised int-soft notions of SS on ordered SGs with the goal of providing a different viewpoint of the interplay between SS and ordered SGs. Khan *et al.,* [21] devised soft union in ordered SGs via uni-soft quasi-ideals. Yousafzai *et al.,* [22] presented the concept of non-associative ordered SGs based on SSs and expanded the existing knowledge about non-associative structures within the framework of SS theory.

Bipolar statements are actually defining the positive and negative aspects of any objects in real life cases. We know that each and every thing in real-life have up to two aspects and these aspects are very useful for any person who wants to knows about benefits and draw begs of any objects. To understand all these aspects in mathematical ways a lot of frameworks are available in market but first of all the idea of bipolar soft theory or bipolar SSs (BSSs) was given by Shabir and Naz [23] introduce the new theory of on bipolar SSs. They also introduce some theory related operations and properties in this manuscript. One another approach for bipolar SSs in different ways are discussed by Karaaslan and Karataş [24] with the title of novel method for treating bipolar SSs and its applications. Moreover, one another and until last approach for bipolar SSs are discussed by Mahmood [25] with the title of a fresh method for treating bipolar SSs and their applications. This approach was very different from other two approaches and cannot compare able with other two approaches. So, the third approach is very advance because it covers all the previous concepts and discussed some new concepts and frameworks. Also, application is available in this manuscript. So, using all these concepts many researchers utilized this idea in different fields and discussed some new concepts and applications. Some useful and necessary ideas are discussed here. Al-Shami [26] discuss the idea of BSSs and connections between them, regular points, and their uses. Kamacı and Petchimuthu [27] also gave the new idea of bipolar N-SST with applications. Ali *et al.,* [28] provided some fresh concepts on bipolar neutrosophic SSs and applications in DM. Shabir *et al.,* [29] extended the idea of BSSs and introduce the new concept of approximate BSSs using soft relations and their application in DM. Saleh *et al.,* [30] gives a very interesting idea by extending the concept of BSSs and introduce binary bipolar soft points and topology on binary BSSs with their symmetric Properties. Moreover, the concept of multiattribute DM under Fermatean fuzzy bipolar soft framework given by Ali and Ansari [31].

1.1 Motivation of the proposed work

The role of SGs is to operate as the tool for the study and modeling of systems and phenomena in which inverse operation is not relevant or necessary. SGs are encountered in the subject of automata theory, coding theory, language theory, and the study of discrete dynamical systems. They also serve as the basis for the study of formal languages, which are helpful in computer science, programming language theory, and other areas. In addition to this, SGs provide a mathematical framework for algebraic study of structures and their properties. Many classes of SGs, e.g., inverse SGs, regular SGs, and completely regular SGs, are well known and have been researched deeply, leading to important results that are widely used. The motivation for SGs results from the fact that not all types of systems or phenomena can be appropriately modeled using the more restrictive structure of groups. SGs provide a framework which is more general and flexible and, therefore, enable one to study an extended spectrum of algebraic structures and their properties, thus, one may have a deeper insight into various mathematical, scientific, and computational problems. So based on these observations, we have explored the notion and properties of the T-Bipolar Soft Semigroup (T-BSSG), which we constructed as a novel approach to T-BSSs. These properties include the AND and OR product, the Restricted Union (Res-Union) and Restricted Intersection (Res-Intersection), and the Extended Union (Ext-Union) and Extended Intersection (Ext-Intersection) on T-BSSG. Further, we also devise the related algebraic properties of T-BSSG.

The arrangement of the manuscript is as follows; Section 2 is about the preliminary definition of SG, subsemigroup, soft set, soft subset and T-bipolar soft set. In section 3, we have developed the basic definition of T-bipolar soft semigroup and their related results. Section 4 is about the conclusion remarks.

2. Preliminaries

This section is devoted to defining some fundamental concepts that can help with further discussion.

Definition 1: A set $\ddot{X} \neq \emptyset$ with binary operation $*$ is interpreted as SG, if the underneath property is satisfied $\forall \eta_1, \eta_2, \eta_3 \in \ddot{X}$, $(\eta_1 * \eta_2) * \eta_3 = \eta_1 * (\eta_2 * \eta_3)$.

Definition 2: Consider $\ddot{X}$ is a SG and $\emptyset \neq \ddot{X}_b \subseteq \ddot{X}$, then $\ddot{X}_b$ is interpreted as subsemigroup of $\ddot{X}$ if $\forall\ \eta_1, \eta_2 \in\ \ddot{X}_b$, $\eta_1\eta_2 \in \ddot{X}_b$.

Definition 3 [4]: Assume, $\hat{\mathrm{R}}$ represent the universal set, with a set of parameter Ꝁ. For any $ƛ \subseteq$ Ꝁ such that $ƛ \neq \emptyset$. Then a function $⅂_1: ƛ \rightarrow P(\hat{\mathrm{R}})$ over $\hat{\mathrm{R}}$ is called a soft set.

Definition 4 [5]: Let $\hat{\mathrm{R}}$ represent the universal set, with the set of parameter E. Such that $ƛ \neq \emptyset$, consider $(⅂_1, ƛ_1)$ and $(⅂_2, ƛ_2)$ be two SSs over $\hat{\mathrm{R}}$, then $(⅂_1, ƛ_1)$ is known as a soft subset of $(⅂_2, ƛ_2)$, if.

$$ƛ_1 \subseteq ƛ_2,$$
$$\forall\, \tilde{\mathrm{n}} \in ƛ_1 \Rightarrow ⅂_1(\tilde{\mathrm{n}}) \subseteq ⅂_2(\tilde{\mathrm{n}}).$$

If $(⅂_1, ƛ_1)$ is a subset of $(⅂_2, ƛ_2)$ and $(⅂_2, ƛ_2)$ is a subset of $(⅂_1, ƛ_1) \Rightarrow (⅂_1, ƛ_1) = (⅂_2, ƛ_2)$ i.e. the SS are equal. If $(⅂_1, ƛ_1)$ is a soft subset of $(⅂_2, ƛ_2)$, then $(⅂_2, ƛ_2)$ is known as a soft super set of $(⅂_1, ƛ_1)$ is denotad by $(⅂_1, ƛ_1) \tilde{\sqsubseteq} (⅂_2, ƛ_2)$.

Definition 5: Let $(⅂, \breve{℉})$ be any SS and let $\ddot{\mathrm{X}}$ be any SG. Then $(⅂, \breve{℉})$ is called SSG over $\ddot{\mathrm{X}}$ iff $⅂(ƛ)$ is subsemigroup of $\ddot{\mathrm{X}}$, $(⅂(ƛ) \leq \ddot{\mathrm{X}})\ \forall\, ƛ \in \breve{℉}$.

Definition 6 [23]: Take $\hat{\mathrm{R}}$ as a the universal set and $\breve{℉} \subseteq$ Ꝁ. Also, $\neg\breve{℉} = \{\neg ʒ, ʒ \in \breve{℉}\}$ signify the NOT set of $\breve{℉}$. Then, the triplet $(⅂_{\mathrm{bp}}, §_{\mathrm{bp}}, \breve{℉})$ is interpreted as a Bipolar SS where $⅂_{\mathrm{bp}}: \breve{℉} \rightarrow P(\hat{\mathrm{R}})$ and $§_{\mathrm{bp}}: \neg\breve{℉} \rightarrow P(\hat{\mathrm{R}})$ and $⅂_{\mathrm{bp}}(ʒ) \cap §_{\mathrm{bp}}(\neg ʒ) = \phi$ (null set).

Definition 7 [24]: Take $\hat{\mathrm{R}}$ as a universal set, and Ꝁ as a set of parameter and $\breve{℉}_1 \subseteq$ Ꝁ, $\breve{℉}_2 \subseteq$ Ꝁ such that $\breve{℉}_1 \cup \breve{℉}_2 =$ Ꝁ, and $\breve{℉}_1 \cap \breve{℉}_2 = \phi$, then the triplet $(⅂_{\mathrm{bp}}, §_{\mathrm{bp}}, \breve{℉})$ is devised as bipolar SS (BSS) over $\hat{\mathrm{R}}$, where $⅂_{\mathrm{bp}}: \breve{℉}_1 \rightarrow P(\hat{\mathrm{R}})$ and $§_{\mathrm{bp}}: \breve{℉}_2 \rightarrow P(\hat{\mathrm{R}})$ with $⅂_{\mathrm{bp}}(ʒ) \cap §_{\mathrm{bp}}(\mathrm{g}(ʒ)) = \phi$, and $\mathrm{g}: \breve{℉}_1 \rightarrow \breve{℉}_2$ is a bijective mapping.

Definition 8 [25]: Take $\hat{\mathrm{R}}$ as a universal set, with the set of parameter Ꝁ $\breve{℉} \subseteq$ Ꝁ, $\ddot{\mathrm{X}} \subset \hat{\mathrm{R}}$ and $Y = \hat{\mathrm{R}} - \ddot{\mathrm{X}}$. A triplet $(\overline{⅂}, \overline{§}, \breve{℉})$ is then considered the T-BSSs over $\hat{\mathrm{R}}$, where $\overline{⅂}$ and $\overline{§}$ are set valued mappings given by $\overline{⅂}: \breve{℉} \rightarrow P\ (\ddot{\mathrm{X}})$ and $\overline{§}: \breve{℉} \rightarrow P\ (Y)$.

Here, we can write $(\overline{⅂}, \overline{§}, \breve{℉}) = \{< ƛ, \overline{⅂}(ƛ), \overline{§}(ƛ): \overline{⅂}(ƛ) \in P(\ddot{\mathrm{X}})$ and $\overline{§}(ƛ) \in P(Y) >\}$. For the sake of simplicity, we write $(\overline{⅂}, \overline{§}, \breve{℉}) = \{< ƛ, \overline{⅂}(ƛ), \overline{§}(ƛ) >\}$.

3. T-Bipolar Soft Semigroup and Related Results

In the following, we talk about the definition of the T-bipolar soft semi group (T-BSSG) and explore the properties of the T-BSS such as AND and OR product, Res-Union and Res-Intersection, Ext-Union and Ext- Intersection on T-BSSG.

3.1. T-Bipolar Soft Semigroup

The fundamental definition of the T-BSSG will be covered in this section, along with several fundamental operational laws, including extended union and intersection, restricted union, and intersection, AND and OR product.

Definition 9: Let G and H be two distinct SG, $\hat{\mathrm{R}} = G \cup H$, then for any set $\breve{℉}$, a T-BSS $< \overline{⅂}, \overline{§}, \breve{℉} >$ is said to be a T-BSSG iff $\overline{⅂}(ƛ) \leq G$ and $\overline{§}(ƛ) \leq H\ \ \forall\ ƛ \in \breve{℉}$.

Example 1: Let us assume,

$G = \{\, ȩ, \eta_1, \eta_2, \eta_3, \eta_4 \,\}$, $H = \{\, ǻ, ɓ, c, \eta, y, z \,\}$, $\hat{\mathrm{R}} = G \cup H$, $\breve{℉} = \{ƛ_1, ƛ_2, ƛ_3, ƛ_4\}$.

$$\overline{⅂}(ƛ_1) = \{ȩ\};\ \overline{⅂}(ƛ_2) = \{ȩ, \eta_2, \eta_3\};\ \overline{⅂}(ƛ_3) = \{ȩ, \eta_1, \eta_2, \eta_3, \eta_4\};\ \overline{⅂}(ƛ_4) = \{\eta_3, \eta_4\}.$$
$$\overline{§}(ƛ_1) = \{ǻ, z\};\ \overline{§}(ƛ_2) = \{ǻ, ɓ, c, \eta\};\ \overline{§}(ƛ_3) = \{ǻ, ɓ, c, \eta, y, z\};$$
$$\overline{§}(ƛ_4) = \{ǻ, ɓ, c, \eta, y\}.$$
$$(\overline{⅂}, \overline{§}, \breve{℉}) = \{< ƛ_1, \{ȩ\}, \{ǻ, z\} >, < ƛ_2, \{ȩ, \eta_2, \eta_3\}, \{ǻ, ɓ, c, \eta\} >,$$
$$< ƛ_3, \{ȩ, \eta_1, \eta_2, \eta_3, \eta_4\}, \{ǻ, ɓ, c, \eta, y, z\} >, < ƛ_4, \{\eta_3, \eta_4\}, \{ǻ, ɓ, c, \eta, y\} >\}.$$

Definition 10: Let, $(\overline{⅂}_1, \overline{§}_1, \breve{℉}), (\overline{⅂}_2, \overline{§}_2, \breve{\mathrm{Y}}) \in$ T-BSSG over $\hat{\mathrm{R}}$. Then, $(\overline{⅂}_1, \overline{§}_1, \breve{℉})$ is known to be a T-BSSG subset of $(\overline{⅂}_2, \overline{§}_2, \breve{\mathrm{Y}})$. If,

i) ${}^{\circ}\breve{\mathrm{F}} \subseteq \breve{\mathrm{Y}}$.

ii) $\forall ƛ \in {}^{\circ}\breve{\mathrm{F}},\ \bar{\urcorner}_1(ƛ) \subseteq \bar{\urcorner}_2(ƛ)\,, \bar{\S}_1(ƛ) \supseteq \bar{\S}_2(ƛ)$.

Example 2: We suppose that the

$$G = \{ ȩ, ɳ_1, ɳ_2, ɳ_3, ɳ_4 \},\quad H = \{ ắ, ɓ, c, ɳ, y, z \},\quad \hat{\mathrm{R}} = G \cup H\,, {}^{\circ}\breve{\mathrm{F}} = \{ƛ_1, ƛ_2\}$$

And

$$\breve{\mathrm{Y}} = \{ƛ_1, ƛ_2, ƛ_3\},\qquad \bar{\urcorner}_1(ƛ_1) = \{ȩ\},\qquad \bar{\urcorner}_1(ƛ_2) = \{ȩ, ɳ_2, ɳ_3\}\ ;$$

$$\bar{\urcorner}_2(ƛ_1) = \{ȩ, ɳ_2, ɳ_3\}\ ;\ \bar{\urcorner}_2(ƛ_2) = \{ȩ, ɳ_1, ɳ_2, ɳ_3, ɳ_4\}\ ;\ \bar{\urcorner}_2(ƛ_3) = \{ɳ_3, ɳ_4\}.$$

$$\bar{\S}_1(ƛ_1) = \{ắ, ɓ, c, ɳ, y\}\ ;\ \bar{\S}_1(ƛ_2) = \{ắ, ɓ, c, ɳ, y, z\},$$

$$\bar{\S}_2(ƛ_1) = \{ắ, ɓ, c, ɳ\}\ ;\ \bar{\S}_2(ƛ_2) = \{ắ, z\}\ ;\ \bar{\S}_2(ƛ_3) = \{ắ, ɓ\}.$$

Then,

i. ${}^{\circ}\breve{\mathrm{F}} \subseteq \breve{\mathrm{Y}}$.

ii. $\forall ƛ \in {}^{\circ}\breve{\mathrm{F}},\ \bar{\urcorner}_1(ƛ) \subseteq \bar{\urcorner}_2(ƛ)\,, \bar{\S}_1(ƛ) \supseteq \bar{\S}_2(ƛ)$.

Remarks 1: Generally, if ${}^{\circ}\breve{\mathrm{F}} \subseteq \breve{\mathrm{Y}}$, it is not necessarily that $(\bar{\urcorner}_1, \bar{\S}_1, {}^{\circ}\breve{\mathrm{F}})$ is interpreted as T-BSSG subset of $(\bar{\urcorner}_2, \bar{\S}_2, \breve{\mathrm{Y}})$. It merely means that $\bar{\urcorner}_1(ƛ_n) \subseteq \bar{\urcorner}_2(ƛ_n)$ and $\bar{\S}_2(ƛ_n) \subseteq \bar{\S}_1(ƛ_n)\ \forall\ ƛ_n \in {}^{\circ}\breve{\mathrm{F}}$.

Theorem 1: If $\bar{\S}_1(ƛ)$ is a S-SG of $\bar{\S}_2(ƛ)$ or $\bar{\S}_2(ƛ)$ is a S-SG of $\bar{\S}_1(ƛ)$ $\forall (ƛ, ƛ) \in {}^{\circ}\breve{\mathrm{F}} \times \breve{\mathrm{Y}}$, then the AND product of $(\bar{\urcorner}_1, \bar{\S}_1, {}^{\circ}\breve{\mathrm{F}})$ and $(\bar{\urcorner}_2, \bar{\S}_2, \breve{\mathrm{Y}})$ on $\hat{\mathrm{R}}$ is a T-BSSG over $\hat{\mathrm{R}}$.

Proof: Assume that $\bar{\S}_1(ƛ)$ is a sub semigroup of $\bar{\S}_2(ƛ)$ or $\bar{\S}_2(ƛ)$ is a S-SG of $\bar{\S}_1(ƛ)$, then in either case $\bar{\S}_1(ƛ) \cup \bar{\S}_2(ƛ)$ is a sub semigroup $\forall (ƛ, ƛ) \in {}^{\circ}\breve{\mathrm{F}} \times \breve{\mathrm{Y}}$. Now, consider the sub-SG $\bar{\urcorner}_1(ƛ) \cap \bar{\urcorner}_2(ƛ)$ $\forall (ƛ, ƛ) \in {}^{\circ}\breve{\mathrm{F}} \times \breve{\mathrm{Y}}$ trivially. Because as we know that the intersection of any number of the sub-SG is again sub-SG. This implies that in each case the AND product of $(\bar{\urcorner}_1, \bar{\S}_1, {}^{\circ}\breve{\mathrm{F}})$ and $(\bar{\urcorner}_2, \bar{\S}_2, \breve{\mathrm{Y}})$ on $\hat{\mathrm{R}}$ is a T-BSSG over $\hat{\mathrm{R}}$.

Example 3: Let us assume that the,

$$G = \{ ȩ, ɳ_1, ɳ_2, ɳ_3, ɳ_4 \}\,, H = \{ ắ, ɓ, c, d, ȩ \}\,,\qquad \hat{\mathrm{R}} = G \cup H\,,$$

$$\bar{\urcorner}_1(ƛ_1) = \{ȩ\}\ ;\ \bar{\urcorner}_1(ƛ_2) = \{ȩ, ɳ_2, ɳ_3\}\ ;$$

$$\bar{\urcorner}_2(ƛ_1) = \{ȩ, ɳ_2, ɳ_3\}\ ;\ \bar{\urcorner}_2(ƛ_2) = \{ȩ, ɳ_1, ɳ_2, ɳ_3, ɳ_4\}\ ;\ \bar{\urcorner}_2(ƛ_3) = \{ȩ, ɳ_3, ɳ_4\}.$$

$$\bar{\S}_1(ƛ_1) = \{ắ, d, ȩ\}\ ;\ \bar{\S}_1(ƛ_2) = \{ắ, ɓ, c\}\ ;$$

$$\bar{\S}_2(ƛ_1) = \{ắ, ɓ, c, d\}\ ;\ \bar{\S}_2(ƛ_2) = \{ắ, ɓ, c, d, ȩ\}\ ;\ \bar{\S}_2(ƛ_3) = \{ắ, ɓ\}.$$

$${}^{\circ}\breve{\mathrm{F}} \times \breve{\mathrm{Y}} = \{ (ƛ_1, ƛ_1), (ƛ_1, ƛ_2), (ƛ_1, ƛ_3), (ƛ_2, ƛ_1), (ƛ_2, ƛ_2), (ƛ_2, ƛ_3)\},$$

AND the product of

$$(\bar{\urcorner}_1, \bar{\S}_1, {}^{\circ}\breve{\mathrm{F}}) \wedge (\bar{\urcorner}_2, \bar{\S}_2, \breve{\mathrm{Y}}) = \{< (ƛ, ƛ); \bar{\urcorner}_1(ƛ) \cap \bar{\urcorner}_2(ƛ)\ ;\ \bar{\S}_1(ƛ) \cup \bar{\S}_2(ƛ) > (ƛ, ƛ) \in {}^{\circ}\breve{\mathrm{F}} \times \breve{\mathrm{Y}}\}$$ then the

$$(\bar{\urcorner}_1, \bar{\S}_1, {}^{\circ}\breve{\mathrm{F}}) \wedge (\bar{\urcorner}_2, \bar{\S}_2, \breve{\mathrm{Y}}) =$$

$$= \begin{cases} (ƛ_1, ƛ_1) \Rightarrow \\ \bar{\urcorner}_1(ƛ_1) \cap \bar{\urcorner}_2(ƛ_1) = \{ȩ\}\ ;\ \bar{\S}_1(ƛ_1) \cup \bar{\S}_2(ƛ_1) = \{ắ, ɓ, c, d, ȩ\} \\ (ƛ_1, ƛ_2) \Rightarrow \\ \bar{\urcorner}_1(ƛ_1) \cap \bar{\urcorner}_2(ƛ_2) = \{ȩ\}\ ;\ \bar{\S}_1(ƛ_1) \cup \bar{\S}_2(ƛ_2) = \{ắ, ɓ, c, d, ȩ\} \\ (ƛ_1, ƛ_3) \Rightarrow \\ \bar{\urcorner}_1(ƛ_1) \cap \bar{\urcorner}_2(ƛ_3) = \{ȩ\}\ ;\ \bar{\S}_1(ƛ_1) \cup \bar{\S}_2(ƛ_3) = \{ắ, ɓ, d, ȩ\} \\ (ƛ_2, ƛ_1) \Rightarrow \\ \bar{\urcorner}_1(ƛ_2) \cap \bar{\urcorner}_2(ƛ_1) = \{ȩ, ɳ_2, ɳ_3\}\ ;\ \bar{\S}_1(ƛ_2) \cup \bar{\S}_2(ƛ_1) = \{ắ, ɓ, c, d\} \\ (ƛ_2, ƛ_2) \Rightarrow \\ \bar{\urcorner}_1(ƛ_2) \cap \bar{\urcorner}_2(ƛ_2) = \{ȩ, ɳ_2, ɳ_3\}\ ;\ \bar{\S}_1(ƛ_2) \cup \bar{\S}_2(ƛ_2) = \{ắ, ɓ, c, d, ȩ\} \\ (ƛ_2, ƛ_3) \Rightarrow \\ \bar{\urcorner}_1(ƛ_2) \cap \bar{\urcorner}_2(ƛ_3) = \{ȩ, ɳ_3\}\ ;\ \bar{\S}_1(ƛ_2) \cup \bar{\S}_2(ƛ_3) = \{ắ, ɓ, c\} \end{cases}$$

Remarks 2: Generally, it is not necessary that the product of two T-BSSG should be T-BSSG.

Example 4: Let $\breve{A} = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}, \bar{y}, \bar{z}\}$ and $\breve{B} = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}$ be two semigroups, then for any sets $\breve{℉}$ and $\breve{Ɣ}$, and $(\bar{7}_1, \bar{§}_1, \breve{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{Ɣ})$ be two T-BSSG over $\hat{R} = \breve{A} \cup \breve{B}$.

Assume $\breve{℉} = \{\bar{\eta}_1, \bar{y}_1, \bar{z}_1\}$ and $\breve{Ɣ} = \{\bar{\text{ǎ}}_1, \bar{\text{ƃ}}_1, \bar{c}_1\}$,

$$\bar{7}_1(\bar{\eta}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{y}\}\,;\ \bar{7}_1(\bar{y}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}, \bar{y}\}\,;\ \bar{7}_1(\bar{z}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}\},$$
$$\bar{7}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}\}\,;\ \bar{7}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}, \bar{y}, \bar{z}\}\,;\ \bar{7}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{z}\},$$
$$\bar{§}_1(\bar{\eta}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}\}\,;\ \bar{§}_1(\bar{y}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}\}\,;\ \bar{§}_1(\bar{z}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\},$$
$$\bar{§}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \bar{§}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}\,;\ \bar{§}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}.$$
$$\breve{℉} \times \breve{Ɣ} = \{(\bar{\eta}_1, \bar{\text{ǎ}}_1), (\bar{\eta}_1, \bar{\text{ƃ}}_1), (\bar{\eta}_1, \bar{c}_1), (\bar{y}_1, \bar{\text{ǎ}}_1), (\bar{y}_1, \bar{\text{ƃ}}_1), (\bar{y}_1, \bar{c}_1), (\bar{z}_1, \bar{\text{ǎ}}_1), (\bar{z}_1, \bar{\text{ƃ}}_1), (\bar{z}_1, \bar{c}_1)\}.$$

AND product of the

$$(\bar{7}_1, \bar{§}_1, \breve{℉}) \wedge (\bar{7}_2, \bar{§}_2, \breve{Ɣ}) = \{< (ƛ, ƛ)\,;\ \bar{7}_1(ƛ) \cap \bar{7}_2(ƛ)\ ;\ \bar{§}_1(ƛ) \cup \bar{§}_2(ƛ) > (ƛ, ƛ) \in \breve{℉} \times \breve{Ɣ}\}.$$

$$(\bar{7}_1, \bar{§}_1, \breve{℉}) \wedge (\bar{7}_2, \bar{§}_2, \breve{Ɣ}) =$$
$$\begin{cases} (\bar{\eta}_1, \bar{\text{ǎ}}_1) \Rightarrow \\ \bar{7}_1(\bar{\eta}_1) \cap \bar{7}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}\}\ ;\ \bar{§}_1(\bar{\eta}_1) \cup \bar{§}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}, \\ (\bar{\eta}_1, \bar{\text{ƃ}}_1) \Rightarrow \\ \bar{7}_1(\bar{\eta}_1) \cap \bar{7}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{y}\}\ ;\ \bar{§}_1(\bar{\eta}_1) \cup \bar{§}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}, \\ (\bar{\eta}_1, \bar{c}_1) \Rightarrow \\ \bar{7}_1(\bar{\eta}_1) \cap \bar{7}_2(\bar{c}_1) = \{\bar{\text{ǎ}}\}\ ;\ \bar{§}_1(\bar{\eta}_1) \cup \bar{§}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}, \\ (\bar{y}_1, \bar{\text{ǎ}}_1) \Rightarrow \\ \bar{7}_1(\bar{y}_1) \cap \bar{7}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}\}\ ;\ \bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{d}, \bar{\text{ȩ}}\}, \\ (\bar{y}_1, \bar{\text{ƃ}}_1) \Rightarrow \\ \bar{7}_1(\bar{y}_1) \cap \bar{7}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}, \bar{y}\}\ ;\ \bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}, \\ (\bar{y}_1, \bar{c}_1) \Rightarrow \\ \bar{7}_1(\bar{y}_1) \cap \bar{7}_2(\bar{c}_1) = \{\bar{\text{ǎ}}\}\ ;\ \bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}, \\ (\bar{z}_1, \bar{\text{ǎ}}_1) \Rightarrow \\ \bar{7}_1(\bar{z}_1) \cap \bar{7}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}\}\ ;\ \bar{§}_1(\bar{z}_1) \cup \bar{§}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}, \\ (\bar{z}_1, \bar{\text{ƃ}}_1) \Rightarrow \\ \bar{7}_1(\bar{z}_1) \cap \bar{7}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}\}\ ;\ \bar{§}_1(\bar{z}_1) \cup \bar{§}_2(\bar{\text{ƃ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\}, \\ (\bar{z}_1, \bar{c}_1) \Rightarrow \\ \bar{7}_1(\bar{z}_1) \cap \bar{7}_2(\bar{c}_1) = \{\bar{\text{ǎ}}\}\ ;\ \bar{§}_1(\bar{z}_1) \cup \bar{§}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}. \end{cases}$$

It is not a T-BSSG because $\bar{§}_1(\bar{y}_1) \cup \bar{§}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ƃ}}, \bar{d}, \bar{\text{ȩ}}\}$ is not a sub-semigroup of $\breve{B}$.

Theorem 2: If $\bar{7}_1(ƛ)$ is a sub-semigroup of $\bar{7}_2(ƛ)$ or $\bar{7}_2(ƛ)$ is a sub-semigroup of $\bar{7}_1(ƛ)\ \forall\ (ƛ, ƛ) \in \breve{℉} \times \breve{Ɣ}$. Then the OR product of $(\bar{7}_1, \bar{§}_1, \breve{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{Ɣ})$ over $\hat{R}$ is a T-BSSG over $\hat{R}$.

Proof: Assume that $\bar{7}_1(ƛ)$ is a sub-semigroup of $\bar{7}_2(ƛ)$ or $\bar{7}_2(ƛ)$ is a sub-SG of $\bar{7}_1(ƛ)$, then in both cases $\bar{7}_1(ƛ) \cup \bar{7}_2(ƛ)$ is a sub-semigroup $(ƛ, ƛ) \in \breve{℉} \times \breve{Ɣ}$. Consider, $\bar{§}_1(ƛ) \cap \bar{§}_2(ƛ)\ \forall\ (ƛ, ƛ) \in \breve{℉} \times \breve{Ɣ}$ is sub-SG trivially, because as we know that any number of intersection of sub-SG is a sub-SG. This implies that in each case the OR product of two T-BSSG $(\bar{\text{ㄱ}}_1, \bar{\S}_1, {}^{\circ}\breve{F})$ and $(\bar{\text{ㄱ}}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ over $\hat{R}$ is a T-BSSG over $\hat{R}$.

Example 5: Let $\ddot{X} = \{\text{ȩ}, \eta_1, \eta_2, \eta_3, \eta_4\}$, $Y = \{\text{ǎ}, \text{ɓ}, c, d, \text{ȩ}\}$, $\hat{R} = \ddot{X} \cup Y$,

$${}^{\circ}\breve{F} = \{\text{ƛ}_1, \text{ƛ}_2\} \quad \text{and} \quad \breve{\text{Ɣ}} = \{\text{ƛ}_1, \text{ƛ}_2, \text{ƛ}_3\}$$

$$\bar{\text{ㄱ}}_1(\text{ƛ}_1) = \{\text{ȩ}, \eta_3, \eta_4\}\,;\ \bar{\text{ㄱ}}_1(\text{ƛ}_2) = \{\text{ȩ}\}\,;$$

$$\bar{\text{ㄱ}}_2(\text{ƛ}_1) = \{\eta_3, \eta_4\}\,;\ \bar{\text{ㄱ}}_2(\text{ƛ}_2) = \{\text{ȩ}, \eta_1, \eta_2, \eta_3, \eta_4\}\,;\ \bar{\text{ㄱ}}_2(\text{ƛ}_3) = \{\text{ȩ}, \eta_2, \eta_3\}.$$

$$\bar{\S}_1(\text{ƛ}_1) = \{\text{ǎ}, d, \text{ȩ}\}\,;\ \bar{\S}_1(\text{ƛ}_2) = \{\text{ǎ}, \text{ɓ}, c, d, \text{ȩ}\}\,;$$

$$\bar{\S}_2(\text{ƛ}_1) = \{\text{ǎ}, \text{ɓ}, c, d\}\,;\ \bar{\S}_2(\text{ƛ}_2) = \{\text{ǎ}, \text{ɓ}, c\}\,;\ \bar{\S}_2(\text{ƛ}_3) = \{\text{ǎ}, \text{ɓ}\}.$$

$${}^{\circ}\breve{F} \times \breve{\text{Ɣ}} = \{(\text{ƛ}_1,\text{ƛ}_1), (\text{ƛ}_1,\text{ƛ}_2), (\text{ƛ}_1,\text{ƛ}_3), (\text{ƛ}_2,\text{ƛ}_1), (\text{ƛ}_2,\text{ƛ}_2), (\text{ƛ}_2,\text{ƛ}_3)\}.$$

OR product of $(\bar{\text{ㄱ}}_1, \bar{\S}_1, {}^{\circ}\breve{F}) \vee (\bar{\text{ㄱ}}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = \{< (\text{ƛ}, \text{ƛ})\,;\ \bar{\text{ㄱ}}_1(\text{ƛ}) \cup \bar{\text{ㄱ}}_2(\text{ƛ})\,;\ \bar{\S}_1(\text{ƛ}) \cap \bar{\S}_2(\text{ƛ}) > (\text{ƛ}, \text{ƛ}) \in {}^{\circ}\breve{F} \times \breve{\text{Ɣ}}\}$.

$$(\bar{\text{ㄱ}}_1, \bar{\S}_1, {}^{\circ}\breve{F}) \vee (\bar{\text{ㄱ}}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = \begin{cases} (\text{ƛ}_1,\text{ƛ}_1) \Rightarrow \\ \bar{\text{ㄱ}}_1(\text{ƛ}_1) \cup \bar{\text{ㄱ}}_2(\text{ƛ}_1) = \{\text{ȩ}, \eta_3, \eta_4\}\,;\ \bar{\S}_1(\text{ƛ}_1) \cap \bar{\S}_2(\text{ƛ}_1) = \{\text{ǎ}, d\}, \\ (\text{ƛ}_1,\text{ƛ}_2) \Rightarrow \\ \bar{\text{ㄱ}}_1(\text{ƛ}_1) \cup \bar{\text{ㄱ}}_2(\text{ƛ}_2) = \{\text{ȩ}, \eta_1, \eta_2, \eta_3, \eta_4\}\,;\ \bar{\S}_1(\text{ƛ}_1) \cap \bar{\S}_2(\text{ƛ}_2) = \{\text{ǎ}\}, \\ (\text{ƛ}_1,\text{ƛ}_3) \Rightarrow \\ \bar{\text{ㄱ}}_1(\text{ƛ}_1) \cup \bar{\text{ㄱ}}_2(\text{ƛ}_3) = \{\text{ȩ}, \eta_2, \eta_3, \eta_4\}\,;\ \bar{\S}_1(\text{ƛ}_1) \cap \bar{\S}_2(\text{ƛ}_3) = \{\text{ǎ}\}, \\ (\text{ƛ}_2,\text{ƛ}_1) \Rightarrow \\ \bar{\text{ㄱ}}_1(\text{ƛ}_2) \cup \bar{\text{ㄱ}}_2(\text{ƛ}_1) = \{\text{ȩ}, \eta_3, \eta_4\}\,;\ \bar{\S}_1(\text{ƛ}_2) \cap \bar{\S}_2(\text{ƛ}_1) = \{\text{ǎ}, \text{ɓ}, c, \}, \\ (\text{ƛ}_2,\text{ƛ}_2) \Rightarrow \\ \bar{\text{ㄱ}}_1(\text{ƛ}_2) \cup \bar{\text{ㄱ}}_2(\text{ƛ}_2) = \{\text{ȩ}, \eta_1, \eta_2, \eta_3, \eta_4\}\,;\ \bar{\S}_1(\text{ƛ}_2) \cap \bar{\S}_2(\text{ƛ}_2) = \{\text{ǎ}, \text{ɓ}, c\}, \\ (\text{ƛ}_2,\text{ƛ}_3) \Rightarrow \\ \bar{\text{ㄱ}}_1(\text{ƛ}_2) \cup \bar{\text{ㄱ}}_2(\text{ƛ}_3) = \{\text{ȩ}, \eta_2, \eta_3\}\,;\ \bar{\S}_1(\text{ƛ}_2) \cap \bar{\S}_2(\text{ƛ}_3) = \{\text{ǎ}, \text{ɓ}\}. \end{cases}$$

Remarks 3: Typically, the OR product of two T-BSSG does not have to be T-BSSG.

Example 6: Let $\breve{A} = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\eta}, \bar{y}, \bar{z}\}$ and $\breve{B} = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}$ be two semigroups, then for any sets ${}^{\circ}\breve{F}$ and $\breve{\text{Ɣ}}$, and $(\bar{\text{ㄱ}}_1, \bar{\S}_1, {}^{\circ}\breve{F})$ and $(\bar{\text{ㄱ}}_2, \bar{\S}_2, \breve{\text{Ɣ}})$ be two T-BSSG over $\hat{R} = \breve{A} \cup \breve{B}$.

Let us assume,

$${}^{\circ}\breve{F} = \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}\ \text{ǎ}nd\ \breve{\text{Ɣ}} = \{\bar{\text{ǎ}}_1, \bar{\text{ɓ}}_1, \bar{c}_1\},$$

$$\bar{\text{ㄱ}}_1(\bar{\alpha}) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{y}\}\,;\ \bar{\text{ㄱ}}_1(\bar{\beta}) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\eta}, \bar{y}\}\,;\ \bar{\text{ㄱ}}_1(\bar{\gamma}) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}\},$$

$$\bar{\text{ㄱ}}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\eta}\}\,;\ \bar{\text{ㄱ}}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\eta}, \bar{y}, \bar{z}\}\,;\ \bar{\text{ㄱ}}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{z}\},$$

$$\bar{\S}_1(\bar{\alpha}) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}\}\,;\ \bar{\S}_1(\bar{\beta}) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}\}\,;\ \bar{\S}_1(\bar{\gamma}) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}\},$$

$$\bar{\S}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \bar{\S}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}\}\,;\ \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ǎ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}.$$

$${}^{\circ}\breve{F} \times \breve{\text{Ɣ}} = \{(\bar{\alpha}, \bar{\text{ǎ}}_1), (\bar{\alpha}, \bar{\text{ɓ}}_1), (\bar{\alpha}, \bar{c}_1), (\bar{\beta}, \bar{\text{ǎ}}_1), (\bar{\beta}, \bar{\text{ɓ}}_1), (\bar{\beta}, \bar{c}_1), (\bar{\gamma}, \bar{\text{ǎ}}_1), (\bar{\gamma}, \bar{\text{ɓ}}_1), (\bar{\gamma}, \bar{c}_1)\}.$$

OR product of $(\bar{\text{ㄱ}}_1, \bar{\S}_1, {}^{\circ}\breve{F}) \vee (\bar{\text{ㄱ}}_2, \bar{\S}_2, \breve{\text{Ɣ}}) = \{< (\text{ƛ}, \text{ƛ})\,;\ \bar{\text{ㄱ}}_1(\text{ƛ}) \cup \bar{\text{ㄱ}}_2(\text{ƛ})\ ;\ \bar{\S}_1(\text{ƛ}) \cap \bar{\S}_2(\text{ƛ}) > (\text{ƛ}, \text{ƛ}) \in {}^{\circ}\breve{F} \times \breve{\text{Ɣ}}\}$.

$$(\bar{\text{ㄱ}}_1, \bar{\S}_1, {}^{\circ}\breve{F}) \vee (\bar{\text{ㄱ}}_2, \bar{\S}_2, \breve{\text{Ɣ}}) =$$

$$\begin{cases}
(\bar{\alpha},\bar{\text{ǎ}}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\alpha}) \cup \bar{ℸ}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ},\bar{y}\} \ ; \ \bar{\S}_1(\bar{\alpha}) \cap \bar{\S}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}\}, \\
(\bar{\alpha},\bar{\text{ɓ}}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\alpha}) \cup \bar{ℸ}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ},\bar{y},\bar{z}\} \ ; \ \bar{\S}_1(\bar{\alpha}) \cap \bar{\S}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c}\}, \\
(\bar{\alpha},\bar{c}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\alpha}) \cup \bar{ℸ}_2(\bar{c}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\bar{y},\bar{z}\} \ ; \ \bar{\S}_1(\bar{\alpha}) \cap \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ǎ}},\bar{c}\}, \\
(\bar{\beta},\bar{\text{ǎ}}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\beta}) \cup \bar{ℸ}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ},\bar{y}\} \ ; \ \bar{\S}_1(\bar{\beta}) \cap \bar{\S}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}}\}, \\
(\bar{\beta},\bar{\text{ɓ}}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\beta}) \cup \bar{ℸ}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ},\bar{y},\bar{z}\} \ ; \ \bar{\S}_1(\bar{\beta}) \cap \bar{\S}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}}\}, \\
(\bar{\beta},\bar{c}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\beta}) \cup \bar{ℸ}_2(\bar{c}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ},\bar{y},\bar{z}\} \ ; \ \bar{\S}_1(\bar{\beta}) \cap \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ǎ}}\}, \\
(\bar{\gamma},\bar{\text{ǎ}}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\gamma}) \cup \bar{ℸ}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ}\} \ ; \ \bar{\S}_1(\bar{\gamma}) \cap \bar{\S}_2(\bar{\text{ǎ}}_1) = \{\bar{\text{ǎ}},\bar{d}\}, \\
(\bar{\gamma},\bar{\text{ɓ}}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\gamma}) \cup \bar{ℸ}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\text{ɳ},\bar{y},\bar{z}\} \ ; \ \bar{\S}_1(\bar{\gamma}) \cap \bar{\S}_2(\bar{\text{ɓ}}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\bar{d}\}, \\
(\bar{\gamma},\bar{c}_1) \Rightarrow \\
\bar{ℸ}_1(\bar{\gamma}) \cup \bar{ℸ}_2(\bar{c}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{z}\} \ ; \ \bar{\S}_1(\bar{\gamma}) \cap \bar{\S}_2(\bar{c}_1) = \{\bar{\text{ǎ}},\bar{c},\bar{d}\}.
\end{cases}$$

It is not a T-BSSG because $\bar{ℸ}_1(\bar{\alpha}) \cup \bar{ℸ}_2(\bar{c}_1) = \{\bar{\text{ǎ}},\bar{\text{ɓ}},\bar{c},\bar{y},\bar{z}\}$ is not a sub-semigroup of $\breve{A}$.

Theorem 3: If $\bar{ℸ}_1(\text{ƛ})$ is a sub-semigroup of $\bar{ℸ}_2(\text{ƛ})$ or $\bar{ℸ}_2(\text{ƛ})$ is a sub-semigroup of $\bar{ℸ}_1(\text{ƛ})$ $\forall$ ƛ $\in$ ℉ $\cup$ Ɣ. Then, the Extended union of two T-BSSG $(\bar{ℸ}_1, \bar{\S}_1, ℉)$ and $(\bar{ℸ}_2, \bar{\S}_2, Ɣ)$ over R̂ is a T-BSSG over R̂.

Proof: Assume that $\bar{ℸ}_1(\text{ƛ})$ is a sub-semigroup of $\bar{ℸ}_2(\text{ƛ})$ or $\bar{ℸ}_2(\text{ƛ})$ is a sub-semigroup of $\bar{ℸ}_1(\text{ƛ})$, then in both cases $\bar{ℸ}_1(\text{ƛ}) \cup \bar{ℸ}_2(\text{ƛ})$ is a sub-semigroup $\forall$ ƛ $\in$ ℉ $\cup$ Ɣ. If ƛ $\in$ ℉ $-$ Ɣ or ƛ $\in$ Ɣ $-$ ℉, then it is obvious. Now, consider $\bar{\S}_1(\text{ƛ}) \cap \bar{\S}_2(\text{ƛ})$ $\forall$ ƛ $\in$ ℉ $\cup$ Ɣ is a sub-SG trivially, because as we have that intersection of any numbers of sub-SG is a sub-SG. This implies that in each case the Ext-U of two T-BSSG $(\bar{ℸ}_1, \bar{\S}_1, ℉)$ and $(\bar{ℸ}_2, \bar{\S}_2, Ɣ)$ over R̂ is a T-BSSG over R̂.

Example 7: Let $G = \{$ ę, ɳ$_1$, ɳ$_2$, ɳ$_3$, ɳ$_4 \}$, $H = \{$ ǎ, ɓ, $c, d,$ ę $\}$, R̂ $= G \cup H$,

℉ $= \{$ƛ$_1$, ƛ$_2\}$ and Ɣ $= \{$ƛ$_1$, ƛ$_2$, ƛ$_3\}$,

$\bar{ℸ}_1(\text{ƛ}_1) = \{$ę, ɳ$_2$, ɳ$_3\}$; $\bar{ℸ}_1(\text{ƛ}_2) = \{$ę$\}$;

$\bar{ℸ}_2(\text{ƛ}_1) = \{$ę, ɳ$_1$, ɳ$_2$, ɳ$_3$, ɳ$_4\}$; $\bar{ℸ}_2(\text{ƛ}_2) = \{$ɳ$_3$, ɳ$_4\}$; $\bar{ℸ}_2(\text{ƛ}_3) = \{$ę, ɳ$_3$, ɳ$_4\}$;

$\bar{\S}_1(\text{ƛ}_1) = \{$ǎ, $c, d,$ ę$\}$; $\bar{\S}_1(\text{ƛ}_2) = \{$ǎ, ɓ, $c, d,$ ę$\}$;

$\bar{\S}_2(\text{ƛ}_1) = \{$ǎ, $d,$ ę$\}$; $\bar{\S}_2(\text{ƛ}_2) = \{$ǎ, $c, d\}$; $\bar{\S}_2(\text{ƛ}_3) = \{$ǎ, ɓ, $c\}$.

Then the Ext-Union of $(\bar{ℸ}_1, \bar{\S}_1, ℉)$ and $(\bar{ℸ}_2, \bar{\S}_2, Ɣ)$ is denoted and defined by $(\bar{ℸ}_1, \bar{\S}_1, ℉) \cup_{\text{ꝁ}} (\bar{ℸ}_2, \bar{\S}_2, Ɣ) = (\bar{ℸ}_3, \bar{\S}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}}$ = ℉ $\cup$ Ɣ.

$$\bar{ℸ}_3(\text{ƛ}) = \begin{cases} \bar{ℸ}_1(\text{ƛ}) & ; \text{if ƛ} \in ℉ - Ɣ, \\ \bar{ℸ}_2(\text{ƛ}) & ; \text{if ƛ} \in Ɣ - ℉, \\ \bar{ℸ}_1(\text{ƛ}) \cup \bar{ℸ}_2(\text{ƛ}) & ; \text{if ƛ} \in ℉ \cap Ɣ. \end{cases}$$

$$\bar{\S}_3(ℷ) = \begin{cases} \bar{\S}_1(ℷ) & ;\text{if } ℷ \in \breve{℉} - \breve{Ɣ}, \\ \bar{\S}_2(ℷ) & ;\text{if } ℷ \in \breve{Ɣ} - \breve{℉}, \\ \bar{\S}_1(ℷ) \cap \bar{\S}_2(ℷ) & ;\text{if } ℷ \in \breve{℉} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{ך}_3(ℷ_1) = \bar{ך}_1(ℷ_1) \cup \bar{ך}_2(ℷ_1) = \{ȩ, ƞ_1, ƞ_2, ƞ_3, ƞ_4\}\, ;\, ℷ_1 \in \breve{℉} \cap \breve{Ɣ}\,,$$
$$\bar{ך}_3(ℷ_2) = \bar{ך}_1(ℷ_2) \cup \bar{ך}_2(ℷ_2) = \{ȩ, ƞ_3, ƞ_4\}\, ;\, ℷ_2 \in \breve{℉} \cap \breve{Ɣ}\,,$$
$$\bar{ך}_3(ℷ_3) = \bar{ך}_2(ℷ_3) = \{ȩ, ƞ_3, ƞ_4\}\, ;\, ℷ_3 \in \breve{Ɣ} - \breve{℉}\,,$$
$$\bar{\S}_3(ℷ_1) = \bar{\S}_1(ℷ_1) \cap \bar{\S}_2(ℷ_1) = \{ẳ, d, ȩ\}\, ;\, ℷ_1 \in \breve{℉} \cap \breve{Ɣ}\,,$$
$$\bar{\S}_3(ℷ_2) = \bar{\S}_1(ℷ_2) \cap \bar{\S}_2(ℷ_2) = \{ẳ, c, d\}\, ;\, ℷ_2 \in \breve{℉} \cap \breve{Ɣ}\,,$$
$$\bar{\S}_3(ℷ_3) = \bar{\S}_2(ℷ_3) = \{ẳ, б, c\}\, ;\, ℷ_3 \in \breve{Ɣ} - \breve{℉}.$$

Remarks 4: It is noted that the Ext-Union of two T-BSSG is not a T-BSSG.

Example 8: Consider, $\breve{A} = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{ƞ}, \bar{y}, \bar{z}\}$ and $\breve{B} = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{d}, \bar{ȩ}\}$ be two semigroups, then for any sets $\breve{℉}$ and $\breve{Ɣ}$, and $(\bar{ך}_1, \bar{\S}_1, \breve{℉})$ and $(\bar{ך}_2, \bar{\S}_2, \breve{Ɣ})$ be two T-BSSG over $\hat{R} = \breve{A} \cup \breve{B}$. Now assume,

$\breve{℉} = \{\alpha, \beta, \gamma\}$ and $\breve{Ɣ} = \{\alpha, \beta, \eta\}$

$$\bar{ך}_1(\alpha) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{y}\}\, ;\, \bar{ך}_1(\beta) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{ƞ}, \bar{y}\}\, ;\, \bar{ך}_1(\gamma) = \{\bar{ẳ}, \bar{б}\}\,,$$
$$\bar{ך}_2(\alpha) = \{\bar{ẳ}, \bar{z}\}\, ;\, \bar{ך}_2(\beta) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{ƞ}, \bar{y}, \bar{z}\}\, ;\, \bar{ך}_2(\eta) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{ƞ}\}\,,$$
$$\bar{\S}_1(\alpha) = \{\bar{ẳ}, \bar{б}, \bar{c}\}\, ;\, \bar{\S}_1(\beta) = \{\bar{ẳ}, \bar{б}\}\, ;\, \bar{\S}_1(\gamma) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{d}\}\,,$$
$$\bar{\S}_2(\alpha) = \{\bar{ẳ}, \bar{d}, \bar{ȩ}\}\, ;\, \bar{\S}_2(\beta) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{d}\}\, ;\, \bar{\S}_2(\eta) = \{\bar{ẳ}, \bar{c}, \bar{d}, \bar{ȩ}\}.$$

Then the Ext-Union of $(\bar{ך}_1, \bar{\S}_1, \breve{℉})$ and $(\bar{ך}_2, \bar{\S}_2, \breve{Ɣ})$ is denoted and given by $(\bar{ך}_1, \bar{\S}_1, \breve{℉}) \cup_{Ҡ} (\bar{ך}_2, \bar{\S}_2, \breve{Ɣ}) = (\bar{ך}_3, \bar{\S}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{℉} \cup \breve{Ɣ}$.

$$\bar{ך}_3(ℷ) = \begin{cases} \bar{ך}_1(ℷ) & ;\text{if } ℷ \in \breve{℉} - \breve{Ɣ}, \\ \bar{ך}_2(ℷ) & ;\text{if } ℷ \in \breve{Ɣ} - \breve{℉}, \\ \bar{ך}_1(ℷ) \cup \bar{ך}_2(ℷ) & ;\text{if } ℷ \in \breve{℉} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{\S}_3(ℷ) = \begin{cases} \bar{\S}_1(ℷ) & ;\text{if } ℷ \in \breve{℉} - \breve{Ɣ}, \\ \bar{\S}_2(ℷ) & ;\text{if } ℷ \in \breve{Ɣ} - \breve{℉}, \\ \bar{\S}_1(ℷ) \cap \bar{\S}_2(ℷ) & ;\text{if } ℷ \in \breve{℉} \cap \breve{Ɣ}. \end{cases}$$

$$\bar{ך}_3(\alpha) = \bar{ך}_1(\alpha) \cup \bar{ך}_2(\alpha) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{y}, \bar{z}\}\, ;\, \alpha \in \breve{℉} \cap \breve{Ɣ}\,,$$
$$\bar{ך}_3(\beta) = \bar{ך}_1(\beta) \cup \bar{ך}_2(\beta) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{ƞ}, \bar{y}, \bar{z}\}\, ;\, \beta \in \breve{℉} \cap \breve{Ɣ},$$
$$\bar{ך}_3(\gamma) = \bar{ך}_1(\gamma) = \{\bar{ẳ}, \bar{б}\}\, ;\gamma \in \breve{℉} - \breve{Ɣ},$$
$$\bar{ך}_3(\eta) = \bar{ך}_2(\eta) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{ƞ}\}\, ;\, \eta \in \breve{Ɣ} - \breve{℉},$$
$$\bar{\S}_3(\alpha) = \bar{\S}_1(\alpha) \cap \bar{\S}_2(\alpha) = \{\bar{ẳ}\}\, ;\, \alpha \in \breve{℉} \cap \breve{Ɣ},$$
$$\bar{\S}_3(\beta) = \bar{\S}_1(\beta) \cap \bar{\S}_2(\beta) = \{\bar{ẳ}, \bar{б}\}\, ;\, \beta \in \breve{℉} \cap \breve{Ɣ},$$
$$\bar{\S}_3(\gamma) = \bar{\S}_1(\gamma) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{d}\}\, ;\, \gamma \in \breve{℉} - \breve{Ɣ},$$
$$\bar{\S}_3(\eta) = \bar{\S}_2(\eta) = \{\bar{ẳ}, \bar{c}, \bar{d}, \bar{ȩ}\}\, ;\, \eta \in \breve{Ɣ} - \breve{℉}.$$

It is not a T-BSSG because,

$\bar{ך}_3(\alpha) = \bar{ך}_1(\alpha) \cup \bar{ך}_2(\alpha) = \{\bar{ẳ}, \bar{б}, \bar{c}, \bar{y}, \bar{z}\}\, ;\, \alpha \in \breve{℉} \cap \breve{Ɣ}$, is not a sub-semigroup of $\breve{A}$.

Theorem 4: If $\bar{\S}_1(ℷ)$ is a sub-semigroup of $\bar{\S}_2(ℷ)$ or $\bar{\S}_2(ℷ)$ is a sub-semigroup of $\bar{\S}_1(ℷ)$ $\forall\, ℷ \in \breve{℉} \cup \breve{Ɣ}$. Then the Ext-Intersection of two T-BSSG $(\bar{ך}_1, \bar{\S}_1, \breve{℉})$ and $(\bar{ך}_2, \bar{\S}_2, \breve{Ɣ})$ over $\hat{R}$ is a T-BSSG over $\hat{R}$.

Proof: Suppose that, if $\bar{\S}_1(ℷ)$ is a sub-semigroup of $\bar{\S}_2(ℷ)$ or $\bar{\S}_2(ℷ)$ is a sub-semigroup of $\bar{\S}_1(ℷ)$, then in both cases $\bar{\S}_1(ℷ) \cup \bar{\S}_2(ℷ)$ is a sub-semigroup $\forall\, ℷ \in \breve{℉} \cup \breve{Ɣ}$. If $ℷ \in \breve{℉} - \breve{Ɣ}$ or $ℷ \in \breve{Ɣ} - \breve{℉}$, then it became a trivial one. Next, we consider $\bar{ך}_1(ℷ) \cap \bar{ך}_2(ℷ)$ $\forall\, ℷ \in \breve{℉} \cup \breve{Ɣ}$ is a sub-SG trivially because as we have that intersection of any numbers of sub-SG is a sub-SG. This implies that in each case the Ext-Intersection of two T-BSSG $(\overline{ꞁ}_1, \overline{§}_1, \breve{°F})$ and $(\overline{ꞁ}_2, \overline{§}_2, \breve{Y})$ over $\hat{R}$ is a T-BSSG over $\hat{R}$.

Example 9: Let, $G = \{ ę, η_1, η_2, η_3, η_4 \}$, $H = \{ ǎ, б, c, d, ę \}$, $\hat{R} = G \cup H$,

$$\breve{°F} = \{ƛ_1, ƛ_2\} \quad \text{and} \quad \breve{Y} = \{ƛ_1, ƛ_2, ƛ_3\},$$
$$\overline{ꞁ}_1(ƛ_1) = \{η_3, η_4\};\ \overline{ꞁ}_1(ƛ_2) = \{ę\};$$
$$\overline{ꞁ}_2(ƛ_1) = \{ę, η_1, η_2, η_3, η_4\};\ \overline{ꞁ}_2(ƛ_2) = \{ę, η_2, η_3\};\ \overline{ꞁ}_2(ƛ_3) = \{ę, η_3, η_4\};$$
$$\overline{§}_1(ƛ_1) = \{ǎ, c, d, ę\};\ \overline{§}_1(ƛ_2) = \{ǎ, б, c, d, ę\};$$
$$\overline{§}_2(ƛ_1) = \{ǎ, d, ę\};\ \overline{§}_2(ƛ_2) = \{ǎ, c, d\};\ \overline{§}_2(ƛ_3) = \{ǎ, б, c\}.$$

Then, the Ext-Intersection of $(\overline{ꞁ}_1, \overline{§}_1, \breve{°F})$ and $(\overline{ꞁ}_2, \overline{§}_2, \breve{Y})$ is denoted and given by $(\overline{ꞁ}_1, \overline{§}_1, \breve{°F}) \cap_{Ҝ} (\overline{ꞁ}_2, \overline{§}_2, \breve{Y}) = (\overline{ꞁ}_3, \overline{§}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{°F} \cup \breve{Y}$.

$$\overline{ꞁ}_3(ƛ) = \begin{cases} \overline{ꞁ}_1(ƛ) & ;\text{if } ƛ \in \breve{°F} - \breve{Y}, \\ \overline{ꞁ}_2(ƛ) & ;\text{if } ƛ \in \breve{Y} - \breve{°F}, \\ \overline{ꞁ}_1(ƛ) \cap \overline{ꞁ}_2(ƛ) & ;\text{if } ƛ \in \breve{°F} \cap \breve{Y}. \end{cases}$$
$$\overline{§}_3(ƛ) = \begin{cases} \overline{§}_1(ƛ) & ;\text{if } ƛ \in \breve{°F} - \breve{Y}, \\ \overline{§}_2(ƛ) & ;\text{if } ƛ \in \breve{Y} - \breve{°F}, \\ \overline{§}_1(ƛ) \cup \overline{§}_2(ƛ) & ;\text{if } ƛ \in \breve{°F} \cap \breve{Y}. \end{cases}$$
$$\overline{ꞁ}_3(ƛ_1) = \overline{ꞁ}_1(ƛ_1) \cap \overline{ꞁ}_2(ƛ_1) = \{η_3, η_4\};\ ƛ_1 \in \breve{°F} \cap \breve{Y},$$
$$\overline{ꞁ}_3(ƛ_2) = \overline{ꞁ}_1(ƛ_2) \cap \overline{ꞁ}_2(ƛ_2) = \{ę\};\ ƛ_2 \in \breve{°F} \cap \breve{Y},$$
$$\overline{ꞁ}_3(ƛ_3) = \overline{ꞁ}_2(ƛ_3) = \{ę, η_3, η_4\};\ ƛ_3 \in \breve{Y} - \breve{°F},$$
$$\overline{§}_3(ƛ_1) = \overline{§}_1(ƛ_1) \cup \overline{§}_2(ƛ_1) = \{ǎ, c, d, ę\};\ ƛ_1 \in \breve{°F} \cap \breve{Y},$$
$$\overline{§}_3(ƛ_2) = \overline{§}_1(ƛ_2) \cup \overline{§}_2(ƛ_2) = \{ǎ, б, c, d, ę\};\ ƛ_2 \in \breve{°F} \cap \breve{Y},$$
$$\overline{§}_3(ƛ_3) = \overline{§}_2(ƛ_3) = \{ǎ, б, c\};\ ƛ_3 \in \breve{Y} - \breve{°F}.$$

Remarks 5: It is noted that the Ext-Intersection of two T-BSSG is not a T-BSSG.

Example 10: Consider, $\breve{A} = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{η}, \bar{y}, \bar{z}\}$ and $\breve{B} = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}, \bar{ę}\}$ be two semigroups, then for any sets $\breve{°F}$ and $\breve{Y}$, and $(\overline{ꞁ}_1, \overline{§}_1, \breve{°F})$ and $(\overline{ꞁ}_2, \overline{§}_2, \breve{Y})$ be two T-BSSG over $\hat{R} = \breve{A} \cup \breve{B}$.

Assume $\breve{°F} = \{\alpha, \beta, \gamma\}$ and $\breve{Y} = \{\alpha, \beta, \eta\}$.

$$\overline{ꞁ}_1(\alpha) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{y}\};\ \overline{ꞁ}_1(\beta) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{η}, \bar{y}\};\ \overline{ꞁ}_1(\gamma) = \{\bar{ǎ}, \bar{б}\},$$
$$\overline{ꞁ}_2(\alpha) = \{\bar{ǎ}, \bar{z}\};\ \overline{ꞁ}_2(\beta) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{η}, \bar{y}, \bar{z}\};\ \overline{ꞁ}_2(\eta) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{η}\},$$
$$\overline{§}_1(\alpha) = \{\bar{ǎ}, \bar{б}\};\ \overline{§}_1(\beta) = \{\bar{ǎ}, \bar{б}, \bar{c}\};\ \overline{§}_1(\gamma) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}\},$$
$$\overline{§}_2(\alpha) = \{\bar{ǎ}, \bar{d}, \bar{ę}\};\ \overline{§}_2(\beta) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{d}\};\ \overline{§}_2(\eta) = \{\bar{ǎ}, \bar{c}, \bar{d}, \bar{ę}\}.$$

Then, the Ext-Intersection of $(\overline{ꞁ}_1, \overline{§}_1, \breve{°F})$ and $(\overline{ꞁ}_2, \overline{§}_2, \breve{Y})$ is denoted and given by $(\overline{ꞁ}_1, \overline{§}_1, \breve{°F}) \cap_{Ҝ} (\overline{ꞁ}_2, \overline{§}_2, \breve{Y}) = (\overline{ꞁ}_3, \overline{§}_3, \breve{\mathcal{R}})$, where $\breve{\mathcal{R}} = \breve{°F} \cup \breve{Y}$.

$$\overline{ꞁ}_3(ƛ) = \begin{cases} \overline{ꞁ}_1(ƛ) & ;\text{if } ƛ \in \breve{°F} - \breve{Y}, \\ \overline{ꞁ}_2(ƛ) & ;\text{if } ƛ \in \breve{Y} - \breve{°F}, \\ \overline{ꞁ}_1(ƛ) \cap \overline{ꞁ}_2(ƛ) & ;\text{if } ƛ \in \breve{°F} \cap \breve{Y}. \end{cases}$$
$$\overline{§}_3(ƛ) = \begin{cases} \overline{§}_1(ƛ) & ;\text{if } ƛ \in \breve{°F} - \breve{Y}, \\ \overline{§}_2(ƛ) & ;\text{if } ƛ \in \breve{Y} - \breve{°F}, \\ \overline{§}_1(ƛ) \cup \overline{§}_2(ƛ) & ;\text{if } ƛ \in \breve{°F} \cap \breve{Y}. \end{cases}$$
$$\overline{ꞁ}_3(\alpha) = \overline{ꞁ}_1(\alpha) \cap \overline{ꞁ}_2(\alpha) = \{\bar{ǎ}\};\ \alpha \in \breve{°F} \cap \breve{Y},$$
$$\overline{ꞁ}_3(\beta) = \overline{ꞁ}_1(\beta) \cap \overline{ꞁ}_2(\beta) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{η}, \bar{y}\};\ \beta \in \breve{°F} \cap \breve{Y},$$
$$\overline{ꞁ}_3(\gamma) = \overline{ꞁ}_1(\gamma) = \{\bar{ǎ}, \bar{б}\};\ \gamma \in \breve{°F} - \breve{Y},$$
$$\overline{ꞁ}_3(\eta) = \overline{ꞁ}_2(\eta) = \{\bar{ǎ}, \bar{б}, \bar{c}, \bar{η}\};\ \eta \in \breve{Y} - \breve{°F},$$

$$\bar{§}_3(\alpha) = \bar{§}_1(\alpha) \cup \bar{§}_2(\alpha) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \alpha \in \text{℉} \cap \breve{\text{Ɣ}}\,,$$
$$\bar{§}_3(\beta) = \bar{§}_1(\beta) \cup \bar{§}_2(\beta) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}\}\,;\ \beta \in \text{℉} \cap \breve{\text{Ɣ}}\,,$$
$$\bar{§}_3(\gamma) = \bar{§}_1(\gamma) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}\}\,;\ \gamma \in \text{℉} - \breve{\text{Ɣ}}\,,$$
$$\bar{§}_3(\eta) = \bar{§}_2(\eta) = \{\bar{\text{ắ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}\,;\ \eta \in \breve{\text{Ɣ}} - \text{℉}.$$

It is not a T-BSSG because,

$\bar{§}_3(\alpha) = \bar{§}_1(\alpha) \cup \bar{§}_2(\alpha) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{d}, \bar{\text{ȩ}}\}$; $\alpha \in \text{℉} \cap \breve{\text{Ɣ}}$, is not a sub-semigroup of $\breve{B}$.

Theorem 5: If $\bar{7}_1(\text{ƛ})$ is a sub-semigroup of $\bar{7}_2(\text{ƛ})$ or $\bar{7}_2(\text{ƛ})$ is a sub-semigroup of $\bar{7}_1(\text{ƛ})$ $\forall\, \text{ƛ} \in \text{℉} \cap \breve{\text{Ɣ}}$. Then, the Res-Union of two T-BSSG $(\bar{7}_1, \bar{§}_1, \text{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}})$ over $\hat{\text{R}}$ is a T-BSSG over $\hat{\text{R}}$.

Proof: Assume that $\bar{7}_1(\text{ƛ})$ is a sub-semigroup of $\bar{7}_2(\text{ƛ})$ or $\bar{7}_2(\text{ƛ})$ is a sub-semigroup of $\bar{7}_1(\text{ƛ})$, $\forall\, \text{ƛ} \in \text{℉} \cap \breve{\text{Ɣ}}$, then in both cases $\bar{7}_1(\text{ƛ}) \cup \bar{7}_2(\text{ƛ})$ is a sub-semigroup. Now, consider $\bar{§}_1(\text{ƛ}) \cap \bar{§}_2(\text{ƛ})$ $\forall\, \text{ƛ} \in \text{℉} \cap \breve{\text{Ɣ}}$ is a sub-SG trivially, because as we have that intersection of any numbers of sub-SG is a sub-SG. This implies that in each case the Res-Union of two T-BSSG $(\bar{7}_1, \bar{§}_1, \text{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}})$ over $\hat{\text{R}}$ is a T-BSSG over $\hat{\text{R}}$.

Theorem 6: If $\bar{§}_1(\text{ƛ})$ is a sub-SG of $\bar{§}_2(\text{ƛ})$ or $\bar{§}_2(\text{ƛ})$ is a sub-SG of $\bar{§}_1(\text{ƛ})$ $\forall\, \text{ƛ} \in \text{℉} \cap \breve{\text{Ɣ}}$. Then the Res-Intersection of two T-BSSG $(\bar{7}_1, \bar{§}_1, \text{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}})$ over $\hat{\text{R}}$ is a T-BSSG over $\hat{\text{R}}$.

Proof: Like Theorem 5.

Example 11: Assume that,

$$G = \{\text{ȩ}, \text{ɳ}_1, \text{ɳ}_2, \text{ɳ}_3, \text{ɳ}_4\}\,, H = \{\text{ắ}, \text{ɓ}, c, d, \text{ȩ}\}\,, \hat{\text{R}} = G \cup H\,,$$
$$\text{℉} = \{\text{ƛ}_1, \text{ƛ}_2\} \quad \text{and} \quad \breve{\text{Ɣ}} = \{\text{ƛ}_1, \text{ƛ}_2, \text{ƛ}_3\},$$
$$\bar{7}_1(\text{ƛ}_1) = \{\text{ɳ}_3, \text{ɳ}_4\}\,;\ \bar{7}_1(\text{ƛ}_2) = \{\text{ȩ}\};$$
$$\bar{7}_2(\text{ƛ}_1) = \{\text{ȩ}, \text{ɳ}_1, \text{ɳ}_2, \text{ɳ}_3, \text{ɳ}_4\}\,;\ \bar{7}_2(\text{ƛ}_2) = \{\text{ȩ}, \text{ɳ}_2, \text{ɳ}_3\}\,;\ \bar{7}_2(\text{ƛ}_3) = \{\text{ȩ}, \text{ɳ}_3, \text{ɳ}_4\};$$
$$\bar{§}_1(\text{ƛ}_1) = \{\text{ắ}, c, d, \text{ȩ}\}\,;\ \bar{§}_1(\text{ƛ}_2) = \{\text{ắ}, \text{ɓ}, c, d, \text{ȩ}\};$$
$$\bar{§}_2(\text{ƛ}_1) = \{\text{ắ}, d, \text{ȩ}\}\,;\ \bar{§}_2(\text{ƛ}_2) = \{\text{ắ}, c, d\}\,;\ \bar{§}_2(\text{ƛ}_3) = \{\text{ắ}, \text{ɓ}, c\}.$$

Then the Res-Union of $(\bar{7}_1, \bar{§}_1, \text{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}})$, is denoted and given by $(\bar{7}_1, \bar{§}_1, \text{℉}) \cup_R (\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}}) = \{<\text{ƛ}, \bar{7}_1(\text{ƛ}) \cup \bar{7}_2(\text{ƛ}),\ \bar{§}_1(\text{ƛ}) \cap \bar{§}_2(\text{ƛ})>;\forall\, \text{ƛ} \in \text{℉} \cap \breve{\text{Ɣ}}\}$.

$$(\bar{7}_1, \bar{§}_1, \text{℉}) \cup_R (\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}}) = \begin{cases} <\ \text{ƛ}_1\,,\ \bar{7}_1(\text{ƛ}_1) \cup \bar{7}_2(\text{ƛ}_1) = \{\text{ȩ}, \text{ɳ}_1, \text{ɳ}_2, \text{ɳ}_3, \text{ɳ}_4\}\,; \\ \bar{§}_1(\text{ƛ}_1) \cap \bar{§}_2(\text{ƛ}_1) = \{\text{ắ}, d, \text{ȩ}\} >;\ \forall\, \text{ƛ}_1 \in \text{℉} \cap \breve{\text{Ɣ}}. \\ <\ \text{ƛ}_2\,,\ \bar{7}_1(\text{ƛ}_2) \cup \bar{7}_2(\text{ƛ}_2) = \{\text{ȩ}, \text{ɳ}_2, \text{ɳ}_3\}\,; \\ \bar{§}_1(\text{ƛ}_2) \cap \bar{§}_2(\text{ƛ}_2) = \{\text{ắ}, c, d\} >;\ \forall\, \text{ƛ}_2 \in \text{℉} \cap \breve{\text{Ɣ}}. \end{cases}$$

"Res-Intersection" of $(\bar{7}_1, \bar{§}_1, \text{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}})$, is denoted and defined by $(\bar{7}_1, \bar{§}_1, \text{℉}) \cap_R (\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}}) = \{<\text{ƛ}, \bar{7}_1(\text{ƛ}) \cap \bar{7}_2(\text{ƛ}),\ \bar{§}_1(\text{ƛ}) \cup \bar{§}_2(\text{ƛ})>;\forall\, \text{ƛ} \in \text{℉} \cap \breve{\text{Ɣ}}\}$.

$$(\bar{7}_1, \bar{§}_1, \text{℉}) \cap_R (\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}}) = \begin{cases} <\ \text{ƛ}_1\,,\ \bar{7}_1(\text{ƛ}_1) \cap \bar{7}_2(\text{ƛ}_1) = \{\text{ɳ}_3, \text{ɳ}_4\}\,; \\ \bar{§}_1(\text{ƛ}_1) \cup \bar{§}_2(\text{ƛ}_1) = \{\text{ắ}, c, d, \text{ȩ}\} >;\ \forall\, \text{ƛ}_1 \in \text{℉} \cap \breve{\text{Ɣ}}. \\ <\ \text{ƛ}_2\,,\ \bar{7}_1(\text{ƛ}_2) \cap \bar{7}_2(\text{ƛ}_2) = \{\text{ȩ}\}\,; \\ \bar{§}_1(\text{ƛ}_2) \cup \bar{§}_2(\text{ƛ}_2) = \{\text{ắ}, \text{ɓ}, c, d, \text{ȩ}\} >;\ \forall\, \text{ƛ}_2 \in \text{℉} \cap \breve{\text{Ɣ}}. \end{cases}$$

Remarks 6: It is noted that Res-Intersection and Res-Union of two T-BSSG over $\hat{\text{R}}$, is not a T-BSSG.

Example 12: Consider, $\breve{A} = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\text{ɳ}}, \bar{y}, \bar{z}\}$ and $\breve{B} = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}, \bar{\text{ȩ}}\}$ be two semigroups, then for any sets ℉ and $\breve{\text{Ɣ}}$, and $(\bar{7}_1, \bar{§}_1, \text{℉})$ and $(\bar{7}_2, \bar{§}_2, \breve{\text{Ɣ}})$ be two T-BSSG over $\hat{\text{R}} = \breve{A} \cup \breve{B}$.

Assume $\text{℉} = \{\alpha, \beta, \gamma\}$ ắnd $\breve{\text{Ɣ}} = \{\alpha, \beta\}$.

$$\bar{7}_1(\alpha) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{y}\}\,;\ \bar{7}_1(\beta) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\text{ɳ}}, \bar{y}\}\,;\ \bar{7}_1(\gamma) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}\}\,,$$
$$\bar{7}_2(\alpha) = \{\bar{\text{ắ}}, \bar{z}\}\,;\ \bar{7}_2(\beta) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{\text{ɳ}}, \bar{y}, \bar{z}\}.$$
$$\bar{§}_1(\alpha) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}\}\,;\ \bar{§}_1(\beta) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}\}\,;\ \bar{§}_1(\gamma) = \{\bar{\text{ắ}}, \bar{\text{ɓ}}, \bar{c}, \bar{d}\}\,,$$

$$\bar{\S}_2(\alpha) = \{\bar{\breve{\text{ā}}}, \bar{d}, \bar{\text{ȩ}}\}\ ;\ \bar{\S}_2(\beta) = \{\bar{\breve{\text{ā}}}, \bar{c}, \bar{d}\}.$$

Then the "Res-Union" of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}})$, is denoted and defined by $(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}}) \cup_R (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}}) = \{< \lambda, \bar{\daleth}_1(\lambda) \cup \bar{\daleth}_2(\lambda), \bar{\S}_1(\lambda) \cap \bar{\S}_2(\lambda) > ; \forall \lambda \in \breve{\text{F}} \cap \breve{\text{Y}}\}$.

$$(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}}) \cup_R (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}}) = \begin{cases} < \alpha\ ,\ \bar{\daleth}_1(\alpha) \cup \bar{\daleth}_2(\alpha) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{c}, \bar{y}, \bar{z}\}\ ; \\ \bar{\S}_1(\alpha) \cap \bar{\S}_2(\alpha) = \{\bar{\breve{\text{ā}}}\} >\ ;\ \forall\, \alpha \in \breve{\text{F}} \cap \breve{\text{Y}}. \\ < \beta\ ,\ \bar{\daleth}_1(\beta) \cup \bar{\daleth}_2(\beta) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}, \bar{y}, \bar{z}\}\ ; \\ \bar{\S}_1(\beta) \cap \bar{\S}_2(\beta) = \{\bar{\breve{\text{ā}}}, \bar{c}\} >\ ;\ \forall\, \beta \in \breve{\text{F}} \cap \breve{\text{Y}}. \end{cases}$$

It is not a T-BSSG because $\bar{\daleth}_1(\alpha) \cup \bar{\daleth}_2(\alpha) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{c}, \bar{y}, \bar{z}\}\ ;\ \forall\, \alpha \in \breve{\text{F}} \cap \breve{\text{Y}}$, is not a sub-semigroup of $\breve{A}$.

"Res-Intersection" of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}})$, is denoted and given by $(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}}) \cap_R (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}}) = \{< \lambda, \bar{\daleth}_1(\lambda) \cap \bar{\daleth}_2(\lambda), \bar{\S}_1(\lambda) \cup \bar{\S}_2(\lambda) > ; \forall \lambda \in \breve{\text{F}} \cap \breve{\text{Y}}\}$.

$$(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}}) \cap_R (\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}}) = \begin{cases} < \ \alpha\ ,\ \bar{\daleth}_1(\alpha) \cap \bar{\daleth}_2(\alpha) = \{\bar{\breve{\text{ā}}}\}\ ; \\ \bar{\S}_1(\alpha) \cup \bar{\S}_2(\alpha) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{d}, \bar{\text{ȩ}}\} >\ ;\ \forall\, \alpha \in \breve{\text{F}} \cap \breve{\text{Y}}. \\ < \ \beta\ ,\ \bar{\daleth}_1(\beta) \cap \bar{\daleth}_2(\beta) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{c}, \bar{\eta}, \bar{y}\}\ ; \\ \bar{\S}_1(\beta) \cup \bar{\S}_2(\beta) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{c}, \bar{d}\} >\ ;\ \forall\, \beta \in \breve{\text{F}} \cap \breve{\text{Y}}. \end{cases}$$

As, $\bar{\S}_1(\alpha) \cup \bar{\S}_2(\alpha) = \{\bar{\breve{\text{ā}}}, \bar{\text{ƃ}}, \bar{d}, \bar{\text{ȩ}}\}\ ;\ \forall\, \alpha \in \breve{\text{F}} \cap \breve{\text{Y}}$, is not a sub-SG of $\breve{B}$. So, due to this reason "Res-Intersection" of $(\bar{\daleth}_1, \bar{\S}_1, \breve{\text{F}})$ and $(\bar{\daleth}_2, \bar{\S}_2, \breve{\text{Y}})$ is not a T-BSSG.

4. Conclusion

Among all the concepts in the framework of mathematical modeling, the most significant and valuable is semigroups (SGs) for the systems for which it is unadvisable or impossible to apply the inverse of operations. They are found in such fields as automata theory and coding theory, linguistics and discrete dynamics and among others. In the recent past, new development in soft set theory has been made and the new concept that was introduced is the T-bipolar soft sets which is a better definition of bipolarity than the previous ones. Building on this foundation, we introduced a pioneering concept: The T-BSSGs. This new framework is developed based on T-SSs and SG theory and gives new prospects for mathematics. Consequently, this paper is aimed at identifying the basic properties of T-BSSGs such as the AND and OR products and the restricted and extended operations. Namely, Res-Union, Res-Intersection, Ext-Union and Ext-Intersection are considered to be particular instances of the described method. It also goes on further to analyse the algebraic properties that are inherently coupled with T-BSSGs. This extensive analysis not only proves the existence and the credibility of the theoretical concept of T-BSSGs but also creates a possibility of the application of the concept in different fields. Therefore, T-BSSGs offer a fresh perspective on the complex structures and allow to analyze them with the aid of T-bipolar soft sets and semigroup theory in case of ineffectiveness of other approaches. These notions can be extended to some complex fuzzy structures given in [32]. These notions can be explored to the notion defined in [33-37].

Author Contributions

Conceptualization, methodology, software, validation, T.M., M.A., U.R., and J.A.; formal analysis, U.R., and J.A.; investigation, resources, data curation, T.M., M.A. and U.R.; writing—original draft preparation, writing—review and editing, T.M., M.A., U.R., and J.A.; visualization, U.R.; supervision, T.A. All authors have read and agreed to the published version of the manuscript.

Funding
This research received no external funding.

Data Availability Statement
The data will be available on reasonable request to corresponding author.

Conflicts of Interest
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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