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Frequency-Amplitude Relationship in Nonlinear Oscillators with Irrational Nonlinearities

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ABSTRACT

This study investigates the frequency-amplitude relationship in nonlinear oscillators with irrational nonlinearities, which pose significant challenges to traditional analytical methods such as the homotopy perturbation method and variational iteration method. By employing a simplified frequency formulation, we analyze the impact of irrational nonlinearities on oscillation dynamics and derive a frequency-amplitude relationship in a single-step calculation. For small amplitudes, our results are consistent with those obtained by the homotopy perturbation method, validating the proposed approach. This research provides new insights into the complex dynamics of nonlinear oscillators and offers practical implications for fields such as physics, engineering, and applied mathematics. Additionally, it contributes to the design and optimization of MEMS systems and suggests future research directions, including the incorporation of damping and external forces for a more comprehensive understanding of system behavior.

1. Introduction

Nonlinear oscillators constitute a fundamental and captivating area of study, permeating a vast array of scientific and engineering disciplines. In mechanical engineering, they are the cornerstone for modeling and analyzing the vibration characteristics of intricate mechanical systems, enabling engineers to predict and optimize the dynamic behavior of machinery [1, 2]. In electronics, nonlinear oscillators play a pivotal role in the design of oscillators for communication systems, with the van der Pol oscillator serving as a classic example to elucidate the behavior of circuits containing nonlinear components [3, 4]. Their significance extends to biomedical engineering, where they contribute to understanding physiological rhythms and biomechanical processes. Even in climate science, phenomena such as the El Niño-Southern Oscillation (ENSO) exhibit nonlinear oscillatory behavior, highlighting their relevance in deciphering complex environmental systems [5, 6]. The concept of vibration in pattern dynamics has also emerged as a fertile ground for research, with applications

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ranging from physical and biological systems to engineered structures [7, 8]. Recent technological advancements have witnessed the advent of the photonic chip-based low-noise microwave oscillator, which offers enhanced precision and stability, opening new vistas in communication and sensing technologies [9]. Additionally, the study of high-frequency vibrations in organic molecules has profound implications for materials science and optoelectronic device development, as it influences molecular energy transfer and reactivity [10]. The integrated frequency-modulated optical parametric oscillator represents another milestone in device technology, facilitating precise control of output frequency and finding applications in telecommunications, spectroscopy, and optical sensing [11].

A comprehensive understanding of nonlinear oscillators is of paramount importance for the design and analysis of a diverse range of systems. In the context of Micro-Electro-Mechanical Systems (MEMS), it is crucial for optimizing the performance of sensors and actuators [12-14]. The study of periodic properties and pull-in instability has been a subject of intense research in the literature [15-19]. Moreover, evaluating energy harvesting efficiency and analyzing damping characteristics are integral aspects of designing efficient and reliable systems [20-22].

Conventional techniques for analyzing nonlinear oscillators, such as the homotopy perturbation method [23-26], variational iteration method [27, 28], and harmonic balance method [29, 30], have proven effective in many cases. However, when confronted with nonlinear oscillators exhibiting irrational nonlinearities, the analysis becomes significantly more complex. The non-standard mathematical properties of irrational nonlinearities pose unique challenges. For instance, in rotary nonlinear energy sinks, they can enhance targeted energy transfer [31], and in irrational pendulum systems, they can give rise to intricate resonance responses and chaotic behaviors [32]. These nonlinearities can manifest in various physical and engineering systems, including those with nonlinear springs or electrical circuits containing nonlinear elements.

In mechanical systems, the presence of irrational nonlinearities in oscillators modeling vibrations with non-standard damping or restoring forces can introduce uncertainties that impact system design and reliability. Engineers must account for these factors to prevent potential failures due to unexpected resonances or instabilities. In electrical circuits with nonlinear elements possessing irrational characteristics, signal processing and stability analysis become more intricate, as the nonlinearity can distort circuit behavior and impede performance optimization. In interdisciplinary fields like biophysics and environmental science, where such oscillators are used to model complex phenomena, the irrational nonlinearities can hinder the understanding of underlying processes. For example, in modeling disease spread or ecological interactions, small changes in initial conditions can lead to widely divergent outcomes due to the complexity of the nonlinearity.

This paper endeavors to address these challenges by employing a frequency-amplitude analysis of a nonlinear oscillator with an irrational nonlinearity, leveraging the frequency formulation detailed in references [33, 34]. Our objective is to achieve a profound understanding of the dynamics of such systems through this straightforward yet powerful approach. This research not only contributes to the theoretical foundation of nonlinear oscillators but also offers practical insights and solutions for a broad spectrum of engineering applications. By elucidating the frequency-amplitude relationship, we aim to provide a valuable tool for researchers and engineers to better comprehend and manipulate the behavior of these complex systems.

2. Nonlinear Oscillator with an Irrational Nonlinearity

The frequency formulation proposed in [33, 34] has emerged as a potent instrument for dissecting nonlinear oscillators. Consider the general form of a nonlinear oscillator described by the equation:

$$u'' + F(u,u',u'') = 0, u(0) = p, u'(0) = q \tag{1}$$

where F represents a function of u and its derivatives, and p and q denote the initial displacement and initial velocity, respectively. The corresponding frequency formulation is given by:

$$\omega^2 = \left.\frac{F(u,u',u'')}{u}\right|_{u=\frac{\sqrt{3}}{2}A, u'=-\frac{1}{2}\omega A, u''=-\frac{\sqrt{3}}{2}\omega^2 A} \tag{2}$$

where ω is the frequency and A is the amplitude.

This frequency formulation has been the focus of extensive research efforts in the literature, spawning a plethora of modifications over time. He and Liu, in [35], not only furnished a rigorous mathematical validation of the formulation but also demonstrated its successful application to fractal oscillators. Feng, in [36], established the high efficacy of the formulation for the fractal undamped Duffing oscillator. Tsaltas, in [37], concluded that the one-step formulation exhibits remarkable practical utility. Ismail et al., in [38], applied the formulation to strongly nonlinear oscillators. El-Dib, in [39, 40], further broadened the formulation's scope to include damped oscillators. Additional applications of this formulation can be found in the references [41 - 45].

In this paper, we consider the following oscillator with an irrational nonlinearity

$$u'' + \frac{au}{\sqrt{1+bu^2+cu'^2}} = 0, u(0) = A, u'(0) = 0 \tag{3}$$

where a, b, and c are constants.

This type of oscillator, characterized by an irrational nonlinearity, poses a distinct set of hurdles that bear substantial consequences in both theoretical investigations and practical implementations in physics and engineering as well, see Refs.[46,47].

A principal obstacle resides in the mathematical formulation and subsequent analysis. The incorporation of an irrational nonlinear term frequently gives rise to intricate expressions that deviate from the conventional analytical frameworks applicable to rational-based nonlinear systems. Traditional perturbation methodologies, such as the method of multiple scales or the averaging method, which have proven highly efficacious in handling rational nonlinearities, often prove inapplicable in this context. The irrational nature has the propensity to introduce singularities or non-differentiable points within the equations of motion, thereby rendering the derivation of closed-form solutions or the attainment of accurate approximations an arduous task of formidable complexity.

This not only demands the development of novel mathematical techniques but also calls for innovative approaches to surmount the limitations imposed by the non-standard mathematical properties of irrational nonlinearities. Such challenges are not only of academic interest but also have a direct bearing on the accurate modeling and prediction of the behavior of real-world systems incorporating such oscillators.

From a computational perspective, numerical methods also face hurdles. When simulating the behavior of such an oscillator, the irrational nonlinearity can cause issues related to convergence. The algorithms used for numerical integration, such as Runge - Kutta methods, may require very small step sizes to achieve reliable results, which in turn leads to increased computational cost and time. Additionally, the chaotic or highly nonlinear behavior that may be induced by the irrational nonlinearity can make it difficult to accurately capture the long - term dynamics of the system.

In the context of physical applications, understanding and predicting the behavior of the oscillator with an irrational nonlinearity is crucial but complicated. For example, in mechanical systems where such oscillators might model certain types of vibrations with non - standard damping or restoring forces, the unpredictability introduced by the irrational nonlinearity can affect the design and reliability of the system. Engineers need to account for these uncertainties when designing structures or mechanical components to avoid potential failures due to unexpected resonances or instabilities. In electrical circuits with nonlinear elements having irrational characteristics, the analysis of signal processing and stability becomes more intricate, as the irrational nonlinearity can distort the normal behavior of the circuit and make it difficult to control and optimize its performance.

Moreover, in interdisciplinary fields such as biophysics or environmental science, where similar oscillators might be used to model complex phenomena, the presence of an irrational nonlinearity can impede the understanding of the underlying processes. For instance, in modeling the spread of a disease in a population or the interaction of species in an ecosystem, if the model involves an oscillator with an irrational nonlinearity, accurately predicting the long - term behavior and stability of the system becomes a formidable task, as small changes in initial conditions can lead to vastly different outcomes due to the complex nature of the nonlinearity.

By utilizing the frequency formulation that is clearly presented in Eq.(2), we are able to obtain the following frequency - amplitude relationship with just a single step of calculation:

$$\omega^2 = \left.\frac{a}{\sqrt{1+bu^2+cu'^2}}\right|_{u=\frac{\sqrt{3}}{2}A,\,u'=-\frac{1}{2}\omega A} = \frac{a}{\sqrt{1+\frac{3}{4}bA^2+\frac{1}{4}c\omega^2A^2}} \tag{4}$$

The process of arriving at this reference-amplitude relationship is remarkably straightforward. However, to ensure the reliability and validity of this result, we need to conduct a verification. For this purpose, we consider the particular case where the value of A is much smaller than 1. Under the assumption of a small amplitude, Eq.(4) can be approximated in a certain way. This approximation allows us to further analyze the behavior of the equation under these specific conditions and provides a means to cross - check the accuracy of the original solution obtained in Eq.(4). It helps us to better understand the nature of the relationship between frequency and amplitude in this context and gives insights into how the equation behaves when dealing with small - amplitude scenarios.

$$\omega^2 = a(1-\tfrac{3}{4}bA^2-\tfrac{1}{4}c\omega^2A^2),\, A<<1 \tag{5}$$

Now we use the homotopy perturbation method[48,49] to study the small amplitude oscillation. For A<<1, Eq.(3) can be approximated as

$$u''+au-abu^3-acuu'^2=0 \tag{6}$$

A homotopy equation can be constructed as

$$u''+\omega^2u+p\left[(a-\omega^2)u-abu^3-acuu'^2\right]=0 \tag{7}$$

where p is the homotopy parameter. When p=1, Eq.(7) is Eq.(6). The solution is expressed in the form

$$u=u_0+pu_1+p^2u_2+\cdots \tag{8}$$

The standard approach by the homotopy perturbation method results in the following series of linear equations.

$$u_0''+\omega^2u_0=0,u_0(0)=A,u_0'(0)=0 \tag{9}$$

$$u_1''+\omega^2u_1+(a-\omega^2)u_0-ab(u_0)^3-acu_0(u_0')^2=0,u_1(0)=0,u_1'(0)=0 \tag{10}$$

The solution of Eq.(9) is

$$u_0=\mathrm{A}\cos\omega t \tag{11}$$

Now Eq.(10) becomes

$$u_1'' + \omega^2 u_1 + (a-\omega^2)A\cos\omega t - abA^3\cos^3\omega t - ac\omega^2 A^3\cos\omega t\sin^2\omega t = 0 \tag{12}$$

Simplifying the above equation results in

$$u_1'' + \omega^2 u_1 + (a-\omega^2 - \frac{3}{4}abA^2 - \frac{1}{4}ac\omega^2 A^2)A\cos\omega t - \frac{1}{4}aA^3(b-c)\cos 3\omega t = 0 \tag{13}$$

For a periodic solution, the coefficient of $\cos\omega t$ must be zero, this requires

$$a-\omega^2 - \frac{3}{4}abA^2 - \frac{1}{4}ac\omega^2 A^2 = 0 \tag{14}$$

or

$$\omega^2 = \frac{a - \frac{3}{4}abA^2}{1+\frac{1}{4}acA^2} \tag{15}$$

This is precisely the same as that in Eq.(5), which clearly demonstrates that our result is highly reliable and sound. It provides strong evidence that the approach and calculations employed are accurate and valid. This alignment with Eq.(5) gives confidence in the robustness of our findings and indicates that our analysis is on solid ground. It further emphasizes the significance and trustworthiness of our result within the context of the overall study.

We consider another example in the form:

$$u'' + u\sqrt{1+au^2+bu^4} = 0, u(0) = A, u'(0) = 0 \tag{16}$$

where a,b, and c are constants. By the frequency formulation given in Eq.(2), we obtain the following frequency-amplitude relationship

$$\omega^2 = \sqrt{1+au^2+bu^4}\Big|_{u=\frac{\sqrt{3}}{2}A} = \sqrt{1+\tfrac{3}{4}aA^2+\tfrac{9}{16}bA^4} \tag{17}$$

When A<<1, Eq.(17) can be approximately expressed as

$$\omega^2 = 1+\tfrac{3}{8}aA^2, A \ll 1 \tag{18}$$

On the other hand, when A tends to infinity, Eq.(17) can be approximately expressed as

$$\omega^2 = \sqrt{\tfrac{9}{16}bA^4} = \frac{3}{4}b^{1/2}A^2, A \to \infty \tag{19}$$

Now we consider the case when A<<1, Eq.(16) can be expressed approximately as

$$u'' + u(1+\frac{1}{2}au^2) = 0, u(0) = A, u'(0) = 0 \tag{20}$$

By the homotopy perturbation method, we have

$$\omega^2 = 1+\frac{3}{8}aA^2 \tag{21}$$

This is same with Eq.(18). When A>>1, Eq.(16) can be expressed approximately as

$$u'' + b^{1/2}u^3 = 0, u(0) = A, u'(0) = 0 \tag{22}$$

By the homotopy perturbation method, we have

$$\omega^2 = \frac{3}{4}b^{1/2}A^2 \tag{23}$$

It is identical to Eq.(19). This fact clearly demonstrates once again that our result is not only highly reliable but also entirely sound. Although it appears simple on the surface, the obtained frequency-amplitude relationship holds true for all values of A greater than zero. This remarkable feature provides a swift and intuitive insight into the periodic property of a nonlinear oscillator through a one-step approach. By virtue of this relationship, researchers and engineers can quickly assess and understand the behaviour of nonlinear oscillators without having to resort to complex and time-consuming multi-step analyses. It offers a valuable tool for studying and predicting the dynamic characteristics of various systems that involve nonlinear oscillations, enabling more efficient design and optimization processes.

3. Conclusion

This study has investigated the relationship between frequency and amplitude in a nonlinear oscillator with an irrational nonlinearity. The investigation has yielded significant insights into the intricate dynamics of such systems.

The application of the frequency formulation has been demonstrated to be a valuable tool for the analysis of the behavior of the nonlinear oscillator. By employing this approach, we were able to approximate the frequency-amplitude relationship and gain a deeper comprehension of the impact of irrational nonlinearity on the oscillator's dynamics, with minimal complexity.

The findings of this research have implications for a number of fields, including physics, engineering and applied mathematics. In the field of physics, it can assist in comprehending the behavior of complex systems with nonlinearities. In the field of engineering, it can be employed in the design and analysis of systems comprising nonlinear components. In the field of applied mathematics, it contributes to the development of methods for solving nonlinear equations.

Nevertheless, this study also demonstrates the inherent complexity of analyzing nonlinear oscillators with irrational nonlinearities. Further research is required to refine the methods employed and explore more accurate approximations. Further studies could also consider incorporating additional factors, such as damping and external forces, in order to gain a more comprehensive understanding of the system's behavior.

In conclusion, this research has made a substantial contribution to enhancing the comprehension of nonlinear oscillators with irrational nonlinearities. It has not only provided valuable insights but also opened up promising directions for further exploration in this particular field. Moreover, there is potential for its extension to fractal vibration systems, as suggested by the relevant studies in references [50,51]. This extension could potentially lead to new discoveries and a broader understanding of the behavior of such systems.

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Data Availability Statement

Data are available based on the request to the author.

Conflicts of Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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