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A Simple Approximation for Indentation of Finite Elastic Layers by Parabolic, Conical, and Flat-ended Cylindrical Indenters

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ABSTRACT

In this work, we develop simple approximation formulas for the indentation of compressible elastic layers of arbitrary thickness by parabolic, conical, and flat-ended cylindrical indenters. Most surprisingly, a compact *"Pythagoras-like"* interpolation between the half-space and thin-layer limits provides an excellent solution to the problem. The proposed empirical expressions are validated against extensive Boundary Element Method simulations, showing remarkable accuracy across a wide range of elastic moduli, Poisson's ratios (up to 1/3), indenter geometries, penetration depths, and layer thicknesses. The approach is further extended to adhesive contacts of flat-ended punches, yielding compact estimates of adhesion forces in layered systems. Owing to their simplicity and robustness, the presented formulas provide practical correction rules for interpreting experimental indentation data, particularly in nanoindentation and atomic force microscopy of thin films and biological materials.

1. Introduction

Indentation methods are an important tool for probing the mechanical properties of elastic layers, coatings, and biological tissues. A critical challenge in interpreting indentation data lies in the influence of finite sample thickness, substrate effects, and boundary conditions, all of which can lead to systematic errors if not properly accounted for. Early theoretical treatments, such as the mathematical analysis of articular cartilage indentation by Hayes *et al.,* [1], provided the first rigorous framework for understanding the role of finite layer thickness in biomechanical contexts. Building on this foundation, Jaffar [2] analyzed frictionless axisymmetric contact between an indenter and an elastic layer, detailing pressure behavior near the edge and the influence of thickness and Poisson's ratio. Chadwick [3] investigated the incompressible thin layer under spherical indentation, showing how confinement alters the edge singularity. More recently, Hensel, McMeeking, and Kossa [4]

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revisited adhesive contact of confined elastic layers, demonstrating how cohesive zone models regularize or replace the classical elastic edge singularity.

Subsequent developments extended these ideas to more general elastic and layered systems. For instance, Hermanowicz [5] presented numerical investigations and approximate formulae for the determination of Young's modulus from force–displacement curves, emphasizing the need for corrections when a rigid substrate underlies the tested material. Aleksandrov [6] derived asymptotic solutions for axisymmetric contact problems in incompressible elastic layers, providing important analytical insights into limiting regimes.

Advances in experimental techniques, particularly atomic force microscopy (AFM), have further highlighted the complexity of interpreting indentation results for thin films and biological materials. Gavara and Chadwick [7] demonstrated AFM-based strategies for determining the elastic moduli of thin samples and adherent cells, while Argatov and Mishuris [8] synthesized indentation testing approaches in biological systems. More recently, AFM-based adhesive indentation of polymer brushes has been analyzed through phenomenological models [9], and controversies surrounding such measurements have been critically discussed [10].

On the modeling side, semi-analytical and reduction methods have proven powerful. Forsbach [11] developed semi-analytical models for bonded and unbonded elastic layers based on the extension of the Method of Dimensionality Reduction (MDR) to layered elastic media by Argatov, Heß, and Popov [12], offering a simple yet accurate approach. Related works have addressed adhesive detachment paradoxes [13] and adhesive contacts between rigid indenters and thin coatings [14]. The predictive capacity of numerical methods has also been demonstrated through experimental verification of boundary element approaches for adhesive contacts of coated half-spaces [15].

While the above literature provides deep theoretical insight, it does not yield simple, closed-form rules that can be readily implemented in engineering practice. The gap is especially evident when attempting to estimate contact stiffness and penetration depth for elastic layers of arbitrary thickness without resorting to elaborate calculations. This motivates the present work, in which we propose a straightforward estimation approach that captures the essential scaling relations between layer thickness, contact radius, and penetration depth, while retaining quantitative accuracy sufficient for engineering use.

2. Indentation of an elastic layer

In the following, we consider a compressible layer with Poisson's ratio ν far from the incompressible limit $\nu = 0.5$. A detailed analysis shows that the effect of transverse contraction plays an essential (qualitative) role starting with approximately $\nu = 0.4$, [11]. Thus, the following consideration is valid under the assumption of $\nu \le 0.33$. It should be noted that the special case $\nu = 0$ can be used for contact with an "unbounded" layer, as was illustrated in [11].

Parabolic indenter

The problem of indentation of an elastic half-space by a parabolic profile

$$z = \frac{r^2}{2R} \tag{1}$$

where r is the in-plane radius and R the radius of curvature of indenter, was solved by H. Hertz in 1882 [16] and is given by the well know equation

$$F_N = \frac{4}{3} E^* R^{1/2} d^{3/2} \tag{2}$$

which is quoted in all classical books on contact mechanics, e.g. [17]. Here E^* is the reduced elastic modulus

$$E^* = \frac{E}{1-\nu^2} \tag{3}$$

E is the Young's modulus and ν the Poisson's ratio of the half-space.

Solution for the indentation of a thin layer with thickness h is also very well known and is given by the equation

$$F_N = \pi \tilde{E} \frac{Rd^2}{h} \tag{4}$$

where $\tilde{E}$ is the modulus of uniaxial compression,

$$\tilde{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}. \tag{5}$$

Derivation of Eq. (4) can be found in numerous sources, e.g. eq. (2.17) in [18].

By indentation of an elastic layer with finite thickness the normal force will transit from the half-space limit (2) for small indentation depths (as long as the contact radius is much smaller than the thickness of the layer, $\sqrt{Rd} \ll h$ to the thin-layer limit (4) in the opposite case of very large contact radius, $\sqrt{Rd} \gg h$.

It seems that there is no simple derivation of the transition between these two limits. In the present paper, we rely on the pure empirical approach and search formally for an approximation of the form

$$F_N = \left[\left(\frac{4}{3} E^* R^{1/2} d^{3/2} \right)^{\alpha} + \left(\pi \tilde{E} \frac{Rd^2}{h} \right)^{\alpha} \right]^{1/\alpha}. \tag{6}$$

The best fit for parameter α should be found by comparison with direct numerical simulation of indentation of elastic layer with a parabolic indenter using the Boundary Element Method described in [19]. Optimization procedure shows that the optimum parameter $\alpha \approx 2$, thus the best approximation is given by the equation

$$F_N = \sqrt{\left(\frac{4}{3} E^* R^{1/2} d^{3/2} \right)^2 + \left(\pi \tilde{E} \frac{Rd^2}{h} \right)^2} \tag{7}$$

This approximation was verified numerically for a wide range of possible parameter values. For visualization, it is convenient to represent the above dependence in dimensionless form:

$$\bar{F}_N = \sqrt{\bar{d}^3 + \left(\bar{d}^2 \bar{R} \right)^2} \tag{8}$$

with

$$\bar{F}_N = \frac{F_N}{\frac{4}{3} E^* R^2}, \quad \bar{d} = \frac{d}{R}, \quad \bar{R} = \frac{R}{h} \frac{3\pi}{4} \frac{(1-\nu)^2}{1-2\nu}, \tag{9}$$

Figure 1 shows the results of comparison of Eq. (8) with BEM simulations. The solid curves represent the approximation (8) and the circles correspond to data points obtained with the BEM described in [19]. In simulation, the following parameters have been used: $E = 0.3\ \mathrm{MPa}$ to $E = 200\ \mathrm{GPa}$, $R = 0.001\ \mathrm{mm}$ to $R = 100\ \mathrm{mm}$, Poisson's ratios were varied within the range $\nu = 0$ to $\nu = 1/3$. The parameters were selected randomly, and the resulting values of h ranged from $2\ \mu\mathrm{m}$ to $200\ \mathrm{mm}$ as a dependent parameter. The excellent agreement between the approximation

(8) and the numerical results demonstrates a remarkable accuracy across arbitrary combinations of elastic moduli, Poisson's ratios, radii, indentation depth and layer thicknesses.

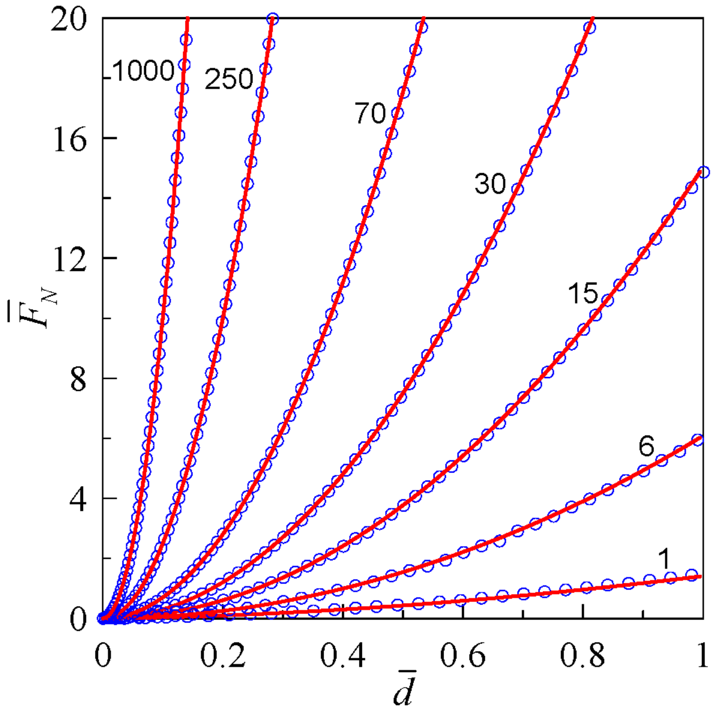

Fig. 1. Comparison of Eq. (8) with result of direct numerical simulations of a parabolic indenter in contact with elastic layer [19]. The numbers at curves denote the value of $\bar{R}$. the following parameters have been used: $E = 0.3$ MPa to $E = 200$ GPa , $R = 0.001$ mm to $R = 100$ mm, Poisson's ratios were varied within the range $\nu = 0$ to $\nu = 1/3$. The parameters were selected randomly, and the resulting values of h ranged from $2\ \mu$m to 200 mm as a dependent parameter; each point in the figure corresponds to a unique random set of parameters.

Conical indenter

Consider indenter in the form of a cone

$$f(r) = r\tan\theta \tag{10}$$

with a small inclination angle θ . Solution for this case was first found by Love [20] (see also [21]):

$$F_N = \frac{2}{\pi}\frac{E^*}{\tan\theta}d^2 \ . \tag{11}$$

The contact force in the limit of a thin layer with thickness h can be easily calculated as

$$F_N = \frac{\tilde{E}}{h}\int_0^{d/\tan\theta}(d - r\tan\theta)2\pi r\mathrm{d}r = \frac{1}{3}\frac{\pi\tilde{E}}{h}\frac{d^3}{\tan^2\theta} \ . \tag{12}$$

If we use a similar form of approximation as (7), we would come to the following general equation

$$F_N = \sqrt{\left(\frac{2}{\pi}\frac{E^*}{\tan\theta}d^2\right)^2 + \left(\frac{1}{3}\frac{\pi\tilde{E}}{h}\frac{d^3}{\tan^2\theta}\right)^2} \ . \tag{13}$$

It can be written in a non-dimensional form

$$\bar{\bar{F}}_N = \sqrt{\bar{\bar{d}}^4 + \kappa\bar{\bar{d}}^6} \tag{14}$$

with

$$\bar{\bar{F}}_N = \frac{\pi F_N}{2E^* h^2 \tan\theta}, \quad \bar{\bar{d}} = \frac{d}{h\tan\theta}, \quad \kappa = \left[\frac{\pi^2}{6}\frac{(1-\nu)^2}{1-2\nu}\right]^2 \tag{15}$$

Figure 2 shows the results of comparison of Eq. (14) with BEM simulations. The excellent agreement between the approximation (14) and the numerical results demonstrates a remarkable accuracy across arbitrary combinations of elastic moduli, Poisson's ratios, inclination angles, indentation depth and layer thicknesses.

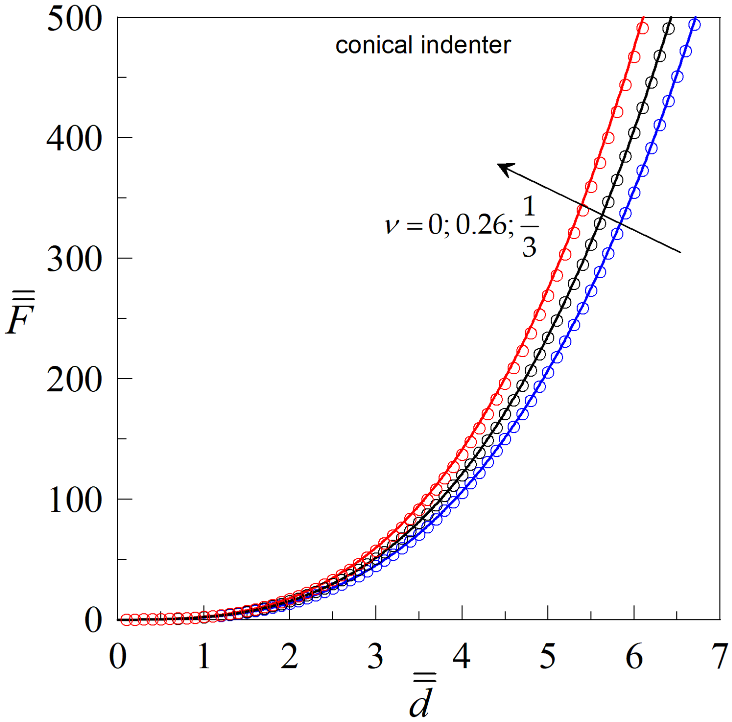

Fig. 2. Comparison of Eq. (14) with result of direct numerical simulations of a conical indenter in contact with BEM described in [19]. The parameter values were chosen randomly in the following ranges: $E = 0.3$ MPa to $E = 200$ GPa, three values of Poisson's ratio: were varied within the range $\nu = 0$, $\nu = 0.26$, and $\nu = 1/3$, The thickness h ranged from $1\ \mu$m to 10 mm, angle θ from 1 to 20 degrees; each point in the figure corresponds to a unique random set of parameters.

Flat-ended cylindrical indenter

Consider indenter in the form of a flat-ended cylinder with radius a. In this case, the contact area does not change during indentation and the normal force is given by the Eq.

$$F_N = K(a)d, \tag{16}$$

where $K(a)$ is the stiffness of the contact. In the limiting case of half-space, the stiffness is given by [18]

$$K(a) = 2aE^*, \tag{17}$$

while for the thin layer it is equal to

$$K(a) = \frac{\pi a^2 \tilde{E}}{h}. \tag{18}$$

Using again the same interpolation rule as above, we can in general case write

$$K(a)=\sqrt{\left(2aE^{*}\right)^{2}+\left(\frac{\pi a^{2}\tilde{E}}{h}\right)^{2}} \tag{19}$$

or in non-dimensional form

$$\hat{K}=\sqrt{\hat{a}^{2}+\beta\hat{a}^{4}}\ , \tag{20}$$

where we introduced notations

$$\hat{K}=\frac{K}{2E^{*}h},\quad \hat{a}=\frac{a}{h},\quad \beta=\left[\frac{\pi}{2}\frac{(1-\nu)^{2}}{1-2\nu}\right]^{2}. \tag{21}$$

Comparison of this extrapolation with BEM simulations is shown in Figure 3.

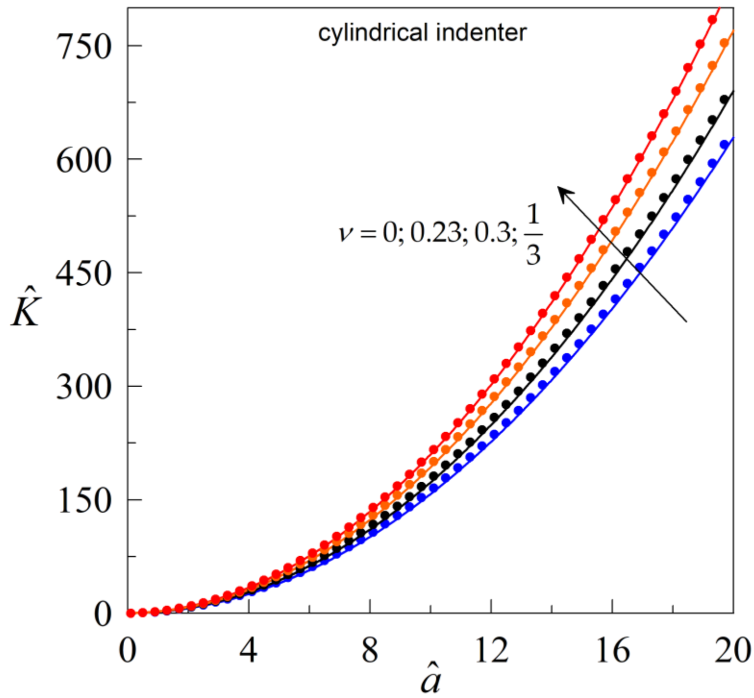

Fig. 3. Comparison of Eq. (20) with result of direct numerical simulations of a conical indenter in contact with a layer using method as described in [19]. The parameter values were chosen randomly in the following ranges: $E=0.3\ \mathrm{MPa}$ to $E=200\ \mathrm{GPa}$, three values of Poisson's ratio: were varied within the range $\nu=0$, $\nu=0.26$, and $\nu=1/3$, the thickness h ranged from $1\ \mu\mathrm{m}$ to $10\ \mathrm{mm}$; each point in the figure corresponds to a unique random set of parameters.

3. Adhesion of a flat-ended cylindrical punch with an elastic layer

Consider an adhesive contact of a flat-ended cylindrical punch of radius a with an elastic layer. It was shown in [22] that the detachment will occur at the critical elongation l_c which satisfies the equation

$$\frac{\mathrm{d}K(a)}{\mathrm{d}a}\frac{l_c^{2}}{2}=2\pi a\gamma \tag{22}$$

where $K(a)$ is stiffness of a non-adhesive contact with radius a. Hence,

$$l_c=\sqrt{\frac{4\pi a\gamma}{\mathrm{d}K(a)/\mathrm{d}a}}\,. \tag{23}$$

For a flat-ended cylindrical punch, the force of adhesion is given by

$$F_{adh} = l_c(a)K(a) = K(a)\sqrt{\frac{4\pi a\gamma}{\mathrm{d}K(a)/\mathrm{d}a}}, \tag{24}$$

or, taking into account equation (19) for stiffness,

$$F_{adh} = \sqrt{8\pi E^* a^3\gamma}\,\frac{\left[1+\left(\dfrac{\pi a\tilde{E}}{2hE^*}\right)^2\right]^{3/4}}{\left[1+2\left(\dfrac{\pi a\tilde{E}}{2hE^*}\right)^2\right]^{1/2}}. \tag{25}$$

This equation can be written in the following non-dimensional form

$$\tilde{\tilde{F}}_{adh} = \frac{\left[1+\tilde{\tilde{a}}^2\right]^{3/4}}{\left[1+2\tilde{\tilde{a}}^2\right]^{1/2}} \tag{26}$$

where

$$\tilde{\tilde{F}}_{adh} = \frac{F_{adh}}{\sqrt{8\pi E^* a^3\gamma}},\quad \tilde{\tilde{a}} = \frac{\pi}{2}\frac{(1-\nu)^2}{1-2\nu}\frac{a}{h}. \tag{27}$$

Figure 4 shows the results of comparison of Eq. (26) with BEM simulations. Comparison demonstrates acceptable accuracy across arbitrary combinations of elastic moduli, Poisson's ratios, punch radii, and layer thicknesses.

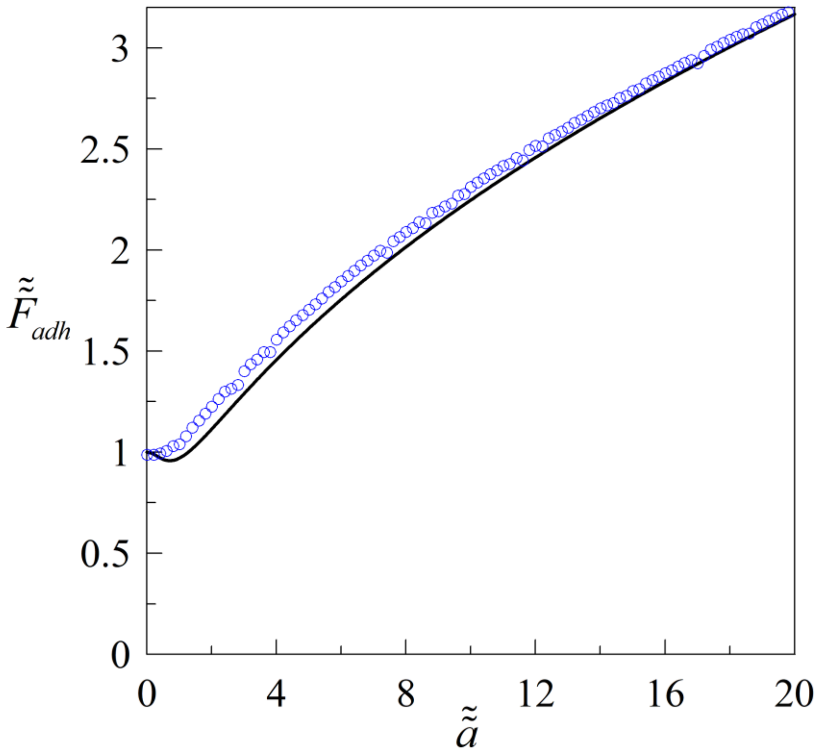

Fig. 4. Comparison of Eq. (26) with result of direct numerical simulations of adhesive contact of a flat-ended punch and an elastic layer using BEM as described in [19]. The parameter values were chosen randomly in the following ranges: $E = 0.3$ MPa to $E = 200$ GPa, three values of Poisson's ratio: were varied within the range $\nu = 0$, $\nu = 0.26$, and $\nu = 0.3$, the thickness h ranged from $1\ \mu$m to 10 mm, work of adhesion $\gamma = 0.01\ \mathrm{J/m^2}$ to $\gamma = 3.0\ \mathrm{J/m^2}$; each point in the figure corresponds to a unique random set of parameters.

4. Discussion

The present work introduces a simple estimation scheme for the indentation of elastic layers of arbitrary thickness, based on the known limiting solutions for half-space and thin layer and a purely empirical optimization procedure on the basis of comparison with "numerically exact" BEM solutions. In the analyzed cases, a simple Pythagorean-type approximation proved to be remarkably effective. Based on these findings, we suggest the following general engineering rule: if the force–indentation relations for the limiting cases of a half-space and a thin layer are given by the equations $F_{\text{half-space}}(d)$ and $F_{\text{thin layer}}(d)$, than the approximation

$$F(d) = \sqrt{\left[F_{\text{half-space}}(d)\right]^2 + \left[F_{\text{thin layer}}(d)\right]^2} \quad (28)$$

provides an excellent engineering approximation for a wide range of indenter shapes, layer thicknesses, and indentation depths, within the practical range of Poisson's ratios between $\nu = 0$ and $\nu = 1/3$. Note that this is correct for both bonded and unbonded layers.

The compact and explicit expression for force-approach dependence provide convenient correction rules for experimental indentation techniques, particularly in atomic force microscopy and micro-/nano-indentation of thin films and biological samples [7]–[10]. In these contexts, neglecting finite-thickness effects leads to systematic overestimation of elastic moduli.

The present formulation opens several promising directions for future research. First, the estimation scheme developed here may be generalized to non-axisymmetric contacts and graded materials, extending the applicability of the method to a wider range of engineering surfaces [23]. Another important direction is the incorporation of poroelastic effects, which are crucial for modeling time-dependent responses of biological tissues such as cartilage [24]. In this context, the simple estimation rules could be coupled with multiphysics approaches to provide compact yet realistic models for hydrated soft layers.

Applications in biomechanics are particularly attractive. Recent *in silicio* studies have demonstrated the role of mechanical stimulation in the regenerative rehabilitation of articular cartilage under cell implantation scenarios [25]. Embedding the present indentation framework into such models may offer a bridge between fundamental contact mechanics and biomedical design of rehabilitation protocols.

5. Conclusion

In this study, we presented a compact approximation scheme for the indentation of finite elastic layers by parabolic, conical, and flat-ended cylindrical indenters. The formulation bridges the gap between rigorous analytical models—which, though exact, are often too cumbersome for routine engineering use—and numerical simulations, which, while highly accurate, can be computationally intensive and difficult to generalize. By combining asymptotically correct limits with empirical optimization against Boundary Element Method simulations, the proposed approximation achieves excellent accuracy across a wide range of layer thicknesses, indenter geometries, elastic moduli, and Poisson's ratios up to 1/3.

Overall, this work shows that carefully constructed approximations can provide powerful tools that unite accuracy, efficiency, and practical applicability. This combination makes the proposed scheme valuable not only in advancing contact mechanics research but also in supporting diverse applications across engineering and applied sciences where indentation testing plays a central role.

Author Contributions

Conceptualization, V.L.P. and M.P.; writing—original draft preparation, V.L.P. and I.L.; writing—review and editing, V.L.P., M.P., and I.L.; visualization, I.L.; supervision, V.L.P.; project administration, M.P. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

This study did not report any data.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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